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STRENGTH OF MATERIALS FOR MECHANICAL ENGINEERS (CE8395)

Unit-1

by

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Course Objectives:

1. To understand the concepts of stress, strain, principal stresses and principal planes.
2. To study the concept of shearing force and bending moment due to external loads in determinate beams and their effect on stresses.
3. To determine stresses and deformation in circular shafts and helical spring due to torsion.
4. To compute slopes and deflections in determinate beams by various methods.
5. To study the stresses and deformations induced in thin and thick shells.

Course outcomes

Students will be able to

1. Understand the concepts of stress and strain in simple and compound bars, the importance of principal stresses and principal planes.
2. Understand the load transferring mechanism in beams and stress distribution due to shearing force and bending moment.
3. Apply basic equation of simple torsion in designing of shafts and helical spring
4. Calculate the slope and deflection in beams using different methods.
5. Analyze and design thin and thick shells for the applied internal and external pressures.

Syllabus Unit-1

UNIT I STRESS, STRAIN AND DEFORMATION OF SOLIDS 9

- Rigid bodies and deformable solids – Tension, Compression and Shear Stresses – Deformation of simple and compound bars – Thermal stresses – Elastic constants – Volumetric strains –Stresses on inclined planes – principal stresses and principal planes – Mohr's circle of stress.

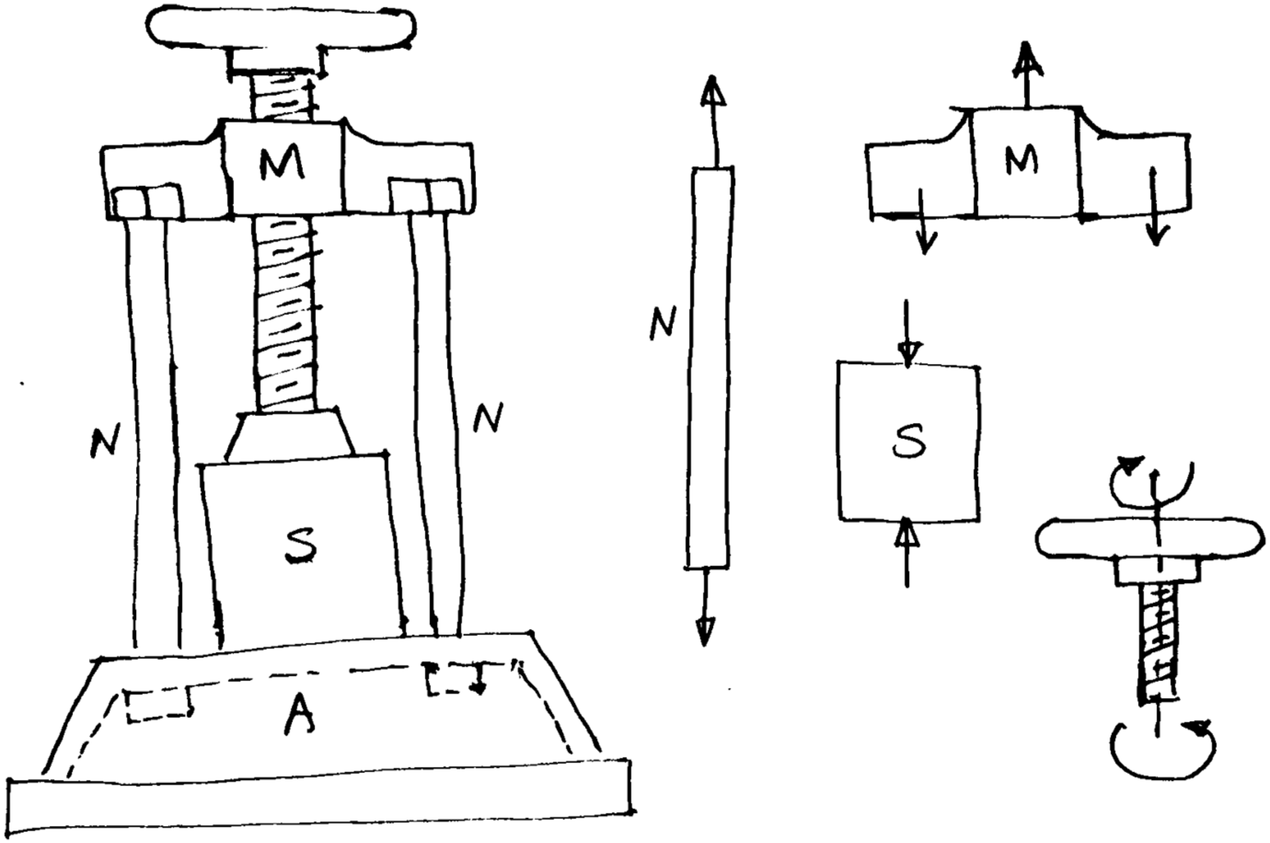
Mechanics

- Rigid body (Engineering mechanics)
- Deformable body (Strength of Materials or mechanics of solids)
- Fluids (Fluid mechanics)

Introduction

- Various structures and machines (eg. Bridges, cranes, airplanes, ships etc.,) consists of numerous parts and members connected together in such a way as to perform a useful function and to withstand externally applied loads
- Consider for example a simple press
- The function of the press is to test specimens of various materials in compression

Introduction



A – Base, S – Specimen, N – Side members
M – Crosshead.

Four basic types of loading

- Tension
- Compression
- Torsion
- Bending

Introduction

- In all engineering construction the component parts must be assigned definite physical sizes
- Such parts should resists the actual or probable forces that may be imposed upon them
- All the above requirements must be met with minimum expenditure of a given material- not only to reduce the cost but also weight
- This subject involves analytical methods for determining 3S's – strength, stiffness and stability

Introduction

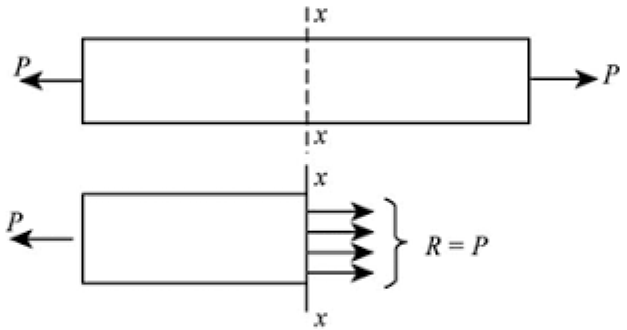
- Tensile force – acts away from the section, tends to elongate the member
- Compressive force – acts towards the bar and tends to compress (shorten) it
- Shearing force – tends to slide or shear the bar
- Twisting moment – tends to twist the bar
- Bending moment – tends to bend the bar

Stress

- The external applied forces on a body tend to deform it and causes it to develop equal and opposite internal forces which resist the deformation
- The resisting force per unit area of the surface is known as stress
- Types of stresses
  - normal stress (tensile and compressive)
  - Shear stress

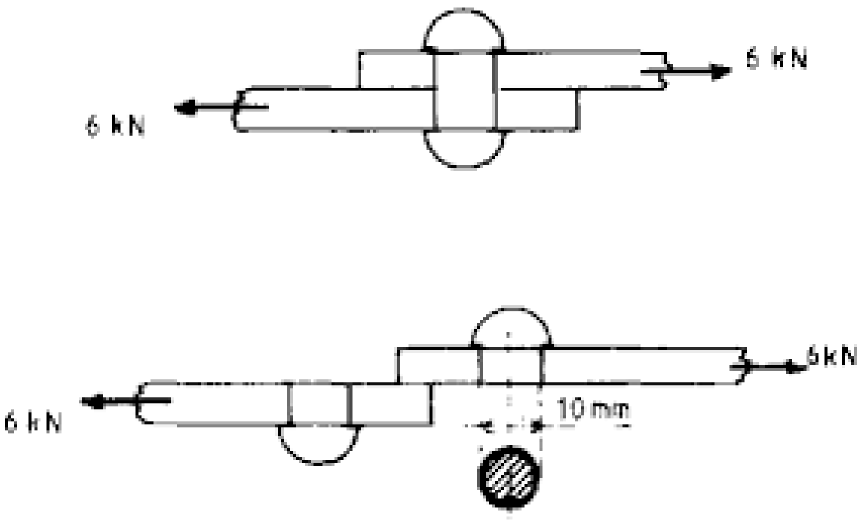
Direct stress (or) normal stress (or) axial stress

- The resulting stress induced, when external forces are applied along the axis



Shear stress

- Simple shear occurs when equal, parallel and opposite forces tend to cause a surface to slide relative to the adjacent surface

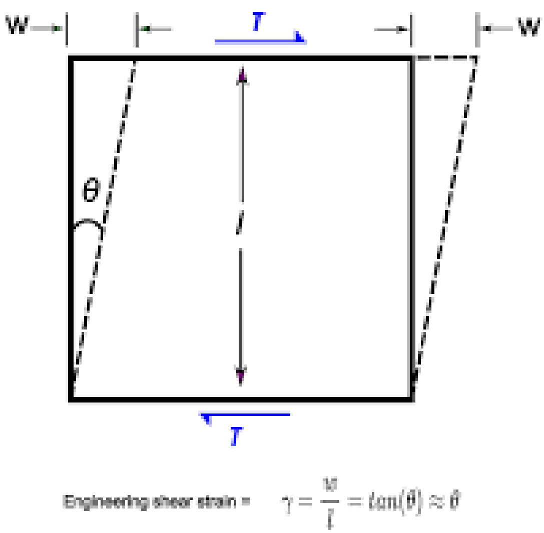


Strain

- Strain – ratio of change in dimension to the original dimension
- Types of strains
  - Normal strain (tensile and compressive)
  - Volumetric strain
  - Shear strain
- Tensile Strain – elongation per unit length

Shear strain

- Shear strain is defined as the change in the right angle measured in radians



Elastic limit

- Elastic material
- Plastic material
- Rigid material
- Elastic limit – the maximum value of stress up to which the material behaves as an elastic material
- Hooke’s law – within elastic limit, the stress is proportional to strain

Elastic Constants

- Young's modulus

$$E = \frac{\sigma}{\varepsilon}$$

- Shear modulus (modulus of rigidity)

$$\mathrm{C}, N \text{ or } G = \frac{\tau}{\varepsilon_s} = \frac{\tau}{\phi}$$

- Bulk modulus

$$K = \frac{\text{normal stress}}{\text{volumetric strain}} = \frac{\sigma_n}{\varepsilon_v}$$

- Poisson's ratio

$$\mu = \frac{\text{lateral or transverse strain}}{\text{linear strain}} = \frac{1}{m}$$

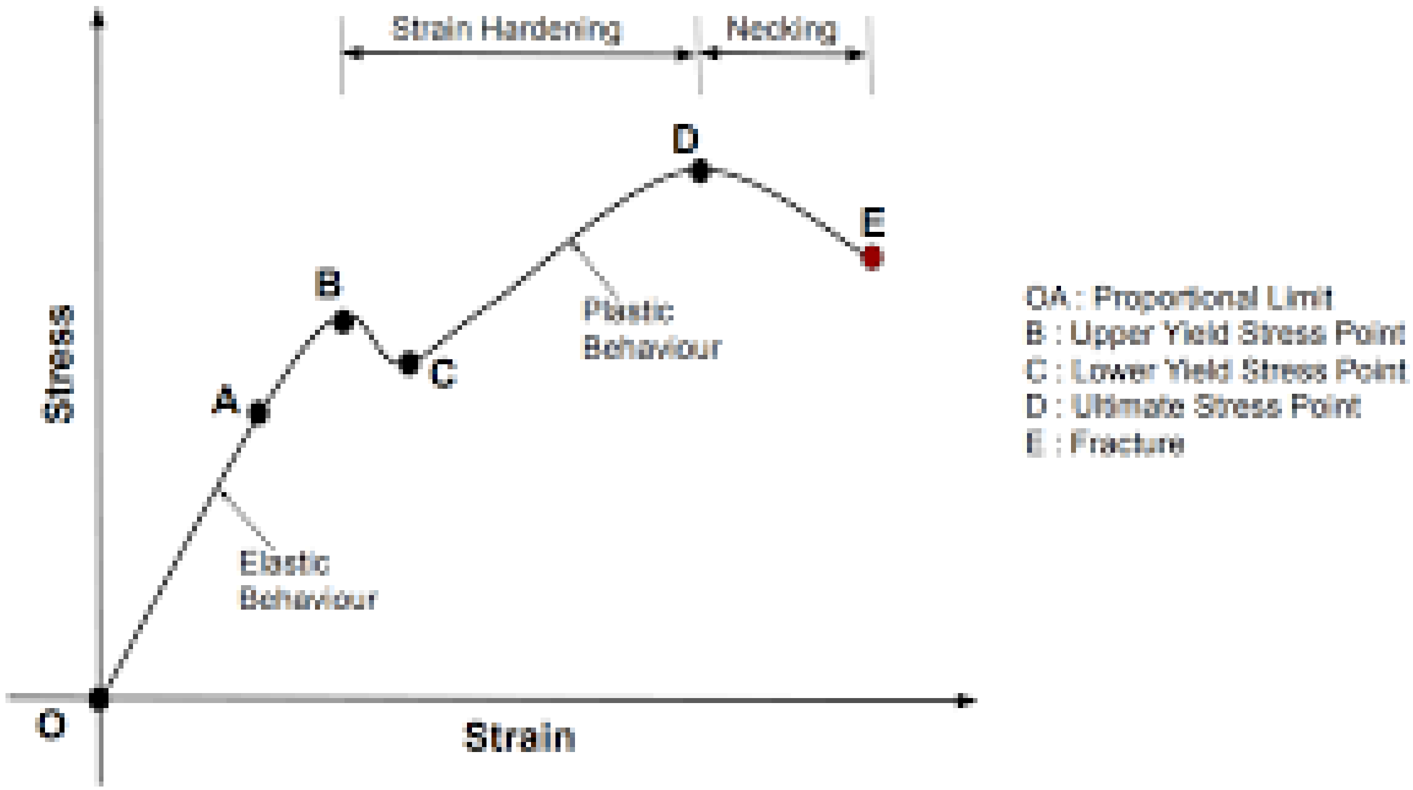
For most materials μ lies in the range 0.25 to 0.35

Some mechanical properties

- **Elasticity** - regain original shape
- **Plasticity** – permanent deformation
- **Ductility** – to be drawn out longitudinally into thin wires
- **Brittleness** – lack of ductility
- **Malleability** – to be drawn into thin sheets under compression
- **Toughness** – ability to absorb energy
- **Hardness** – ability to resist indentation

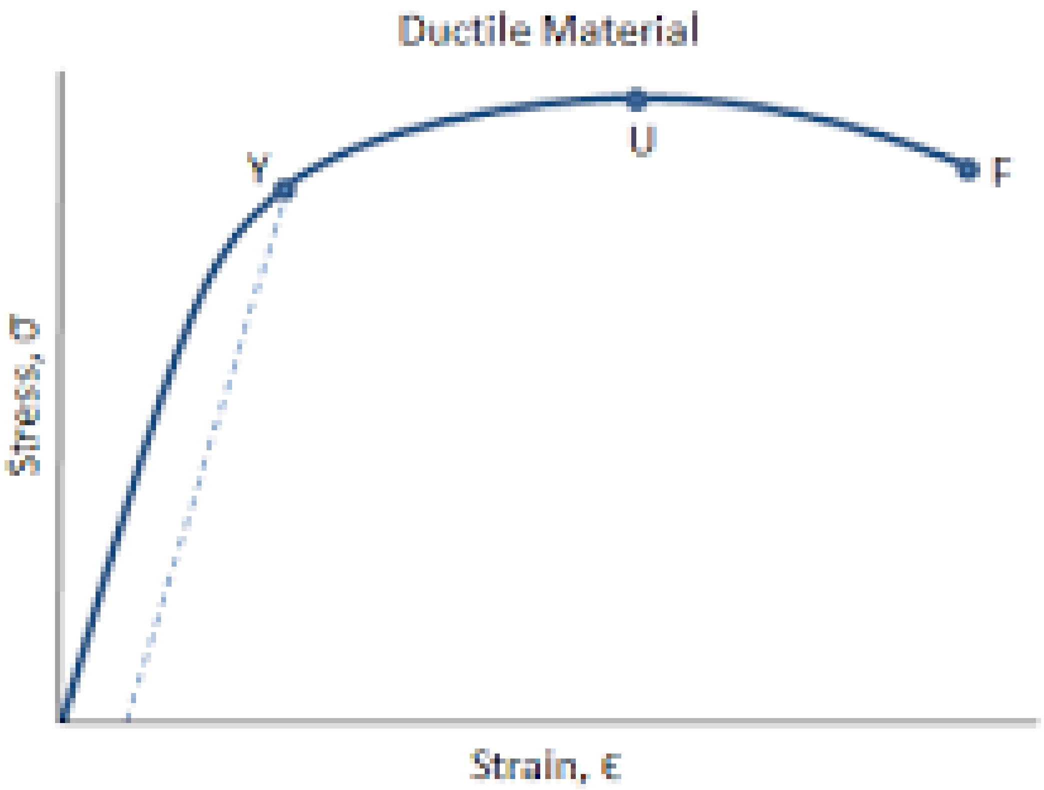
Stress-strain diagram

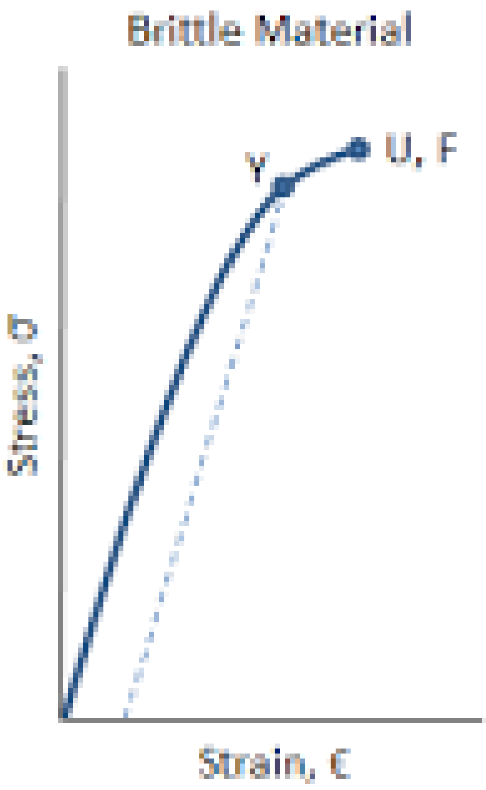
- For ductile materials (Eg. Mildsteel, aluminium, copper, brass, bronze, nylon, Teflon)



Stress-strain diagram

- Brittle materials (cast-iron, concrete, stone, glass, ceramic)





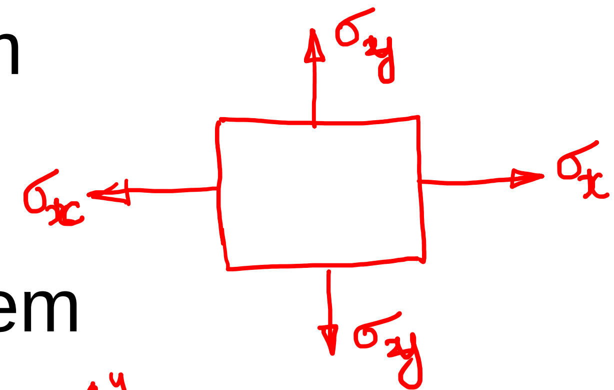
Factor of safety

- **Ultimate stress** - Ultimate load is defined as maximum load which can be placed prior to the breaking of the specimen. Stress corresponding to the ultimate load is known as ultimate stress.
- **Working stress** - the maximum value of stress to which an actual member is expected to be loaded under working condition
- **Factor of safety (F.S.)**
- $F.S. = \frac{ultimate\ stress\ (or) yield\ stress}{working\ stress}$
- For ductile – ultimate stress and for brittle – yield stress

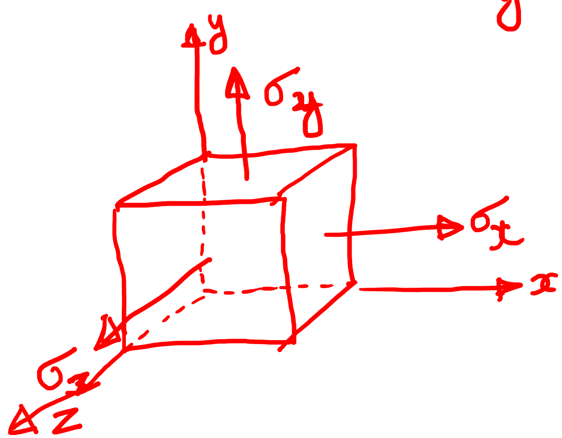
Constitutive relationship between stress and strain

- One dimensional stress system
- $\frac{\sigma}{\varepsilon} = E \; ; \; \varepsilon = \frac{\sigma}{E}$

- Two dimensional stress system
- $\varepsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} \; ; \; \varepsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E}$

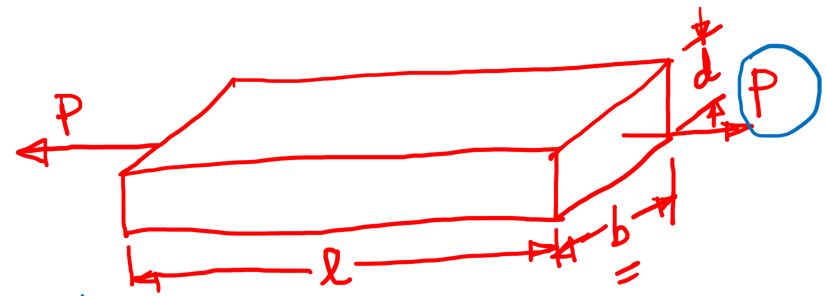


- Three dimensional stress system
- $\varepsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E};$
- $\varepsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E};$
- $\varepsilon_z = \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$



Volumetric strain of a rectangular bar

$$\varepsilon_v = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$$



$V = l\,b\,d$

change in volume = Final vol − original vol

$l + \delta l$

$b - \delta b$

$d - \delta d$

$$\delta V = (l+\delta l)(b-\delta b)(d-\delta d) - lbd$$

$\delta l \times \delta b \to 0$

$\delta b \times \delta d \to 0$

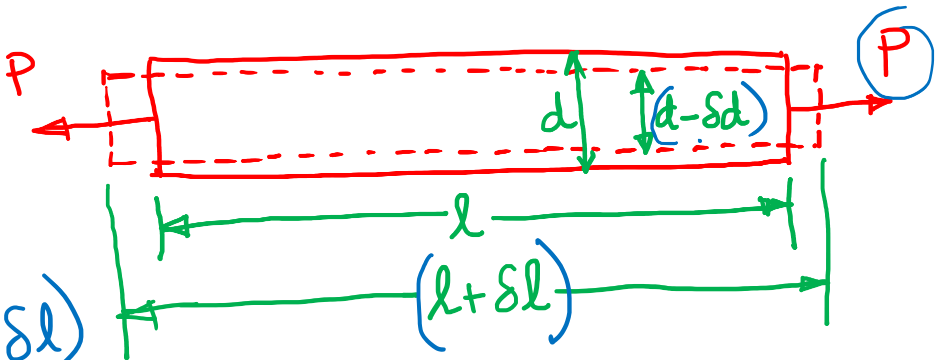
$\delta l \times \delta d \to 0$

$$\delta V = bd\,\delta l - ld\,\delta b - lb\,\delta d$$

$$\varepsilon_v = \frac{bd\,\delta l - ld\,\delta b - lb\,\delta d}{lbd}$$

$$\varepsilon_v = \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta d}{d} = \frac{\delta l}{l} - \mu\frac{\delta l}{l} - \mu\frac{\delta l}{l} = \frac{\delta l}{l}(1-2\mu)$$

Volumetric strain of a cylindrical rod



$$\varepsilon_v = \frac{\delta V}{V}$$

$$V = \frac{\pi}{4} d^2 l$$

$$\text{Final volume} = \frac{\pi}{4}(d-\delta d)^2 (l+\delta l)$$

$$= \frac{\pi}{4}\left(d^2 - 2d\,\delta d + \cancel{\delta d^2}\right)(l+\delta l)$$

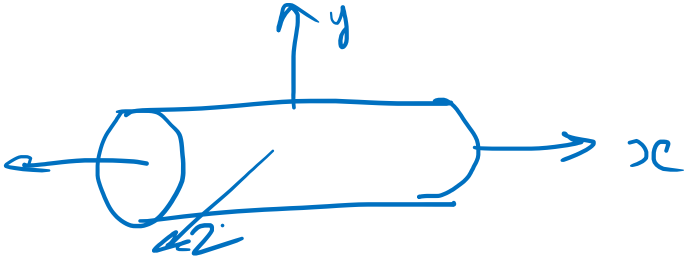
$$= \frac{\pi}{4}\left(d^2 l - 2dl\,\delta d + d^2\delta l - \cancel{2d\,\delta d\,\delta l}\right)$$

$$= \frac{\pi}{4}\left(ld^2 - 2ld\,\delta d + d^2\delta l\right)$$

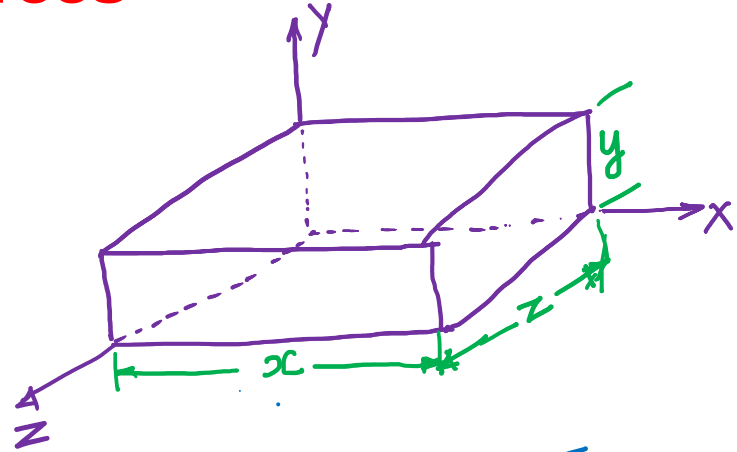
$$\delta V = \frac{\pi}{4}\left(ld^2 - 2ld\,\delta d + d^2\delta l\right) - \frac{\pi}{4}d^2 l = \frac{\pi}{4}\left(d^2\delta l - 2ld\,\delta d\right)$$

$$\varepsilon_v = \frac{\delta V}{V} = \frac{\frac{\pi}{4}\left(d^2 \delta l - 2 l d\, \delta d\right)}{\frac{\pi}{4} d^2 l} = \frac{\delta l}{l} - 2\frac{\delta d}{d}.$$

$$\boxed{\varepsilon_v = \frac{\delta l}{l} - 2\frac{\delta d}{d}}$$



Volumetric strain of a rectangular bar subjected to three mutually perpendicular forces



Volume, $V = xyz$

Taking log on both side

$$\log V = \log x + \log y + \log z$$

Differentiating on both sides

$$\frac{1}{V} dV = \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz.$$

$$\frac{dV}{V} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}.$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z.$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{Z}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$$

$$\boxed{\varepsilon_V = \frac{1}{E}(\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu)}$$

Relation between elastic constants

- Four elastic constants – E, G, K and μ

Home work

1. Derive the relation between E, G and μ
2. Derive the relation between E, K and μ
3. Derive the relation between E, K and G

Relation between elastic constants

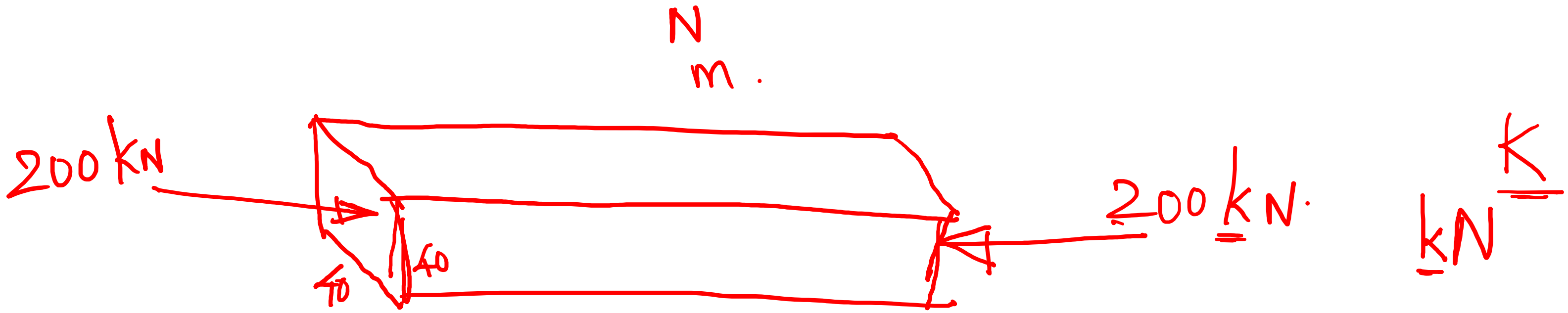
- Four elastic constants – E, G, K and μ

Relation between E, G, K, μ

- $E = 3K(1 - 2\mu)$
- $E = 2G(1 + \mu)$
- $E = \dfrac{9KG}{3K + G}$
- $\mu = \dfrac{3K - 2G}{6K + 2G}$

Here, E = Young's modulus, G = shear modulus
K = Bulk modulus, μ = Poisson ratio

- Bansal, R.K., “Strength of Materials”, Laxmi Publications (P) Ltd.,



4. A steel bar of 40 mm × 40 mm square cross-section is subjected to an axial compressive load of 200 kN. If the length of the bar is 2 m and E = 200 GPa, the elongation of the bar will be:

(a) 1.25 mm

(b) 2.70 mm

(c) 4.05 mm

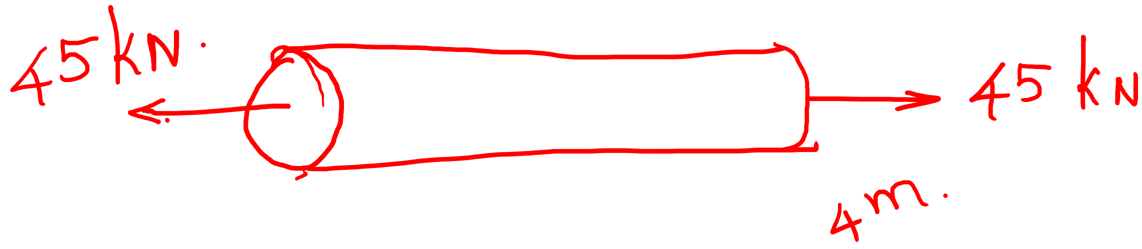
(d) 5.40 mm

Answer

Option (a) is correct

$$\delta = \frac{PL}{AE} = \frac{200 * 1000 * 2}{0.04 * 0.04 * 200 * 10^9} m = 1.25 mm$$

Problems



- A steel rod of 25 mm in diameter and 4 long is subjected to an axial pull of 45 kN. Find -(i) intensity of stress, (ii) the strain, and (iii) the elongation. Take E =2.1 x10^8 kN/m^2.

i) $\sigma = \frac{P}{A} = \frac{45 \times 10^3}{\frac{\pi}{4}(0.025)^2} = 91.668 \times 10^6 \ N/m^2$.

$E = \frac{\sigma}{\varepsilon}$

ii) $\varepsilon = \frac{\sigma}{E} = \frac{91.668 \times 10^6}{2.1 \times 10^8 \times 10^3} = 0.0004365$

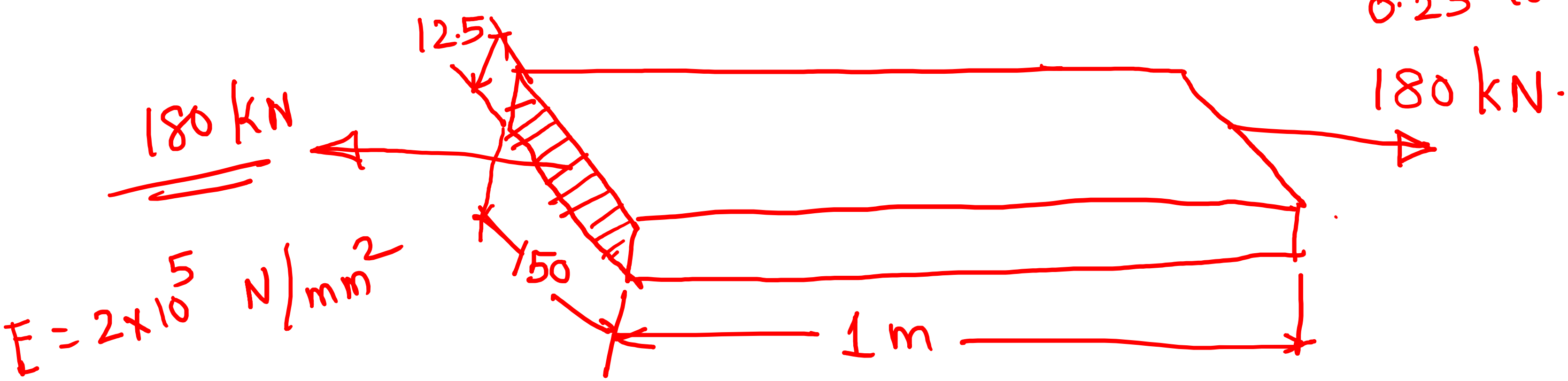
$\varepsilon = \frac{\delta l}{l}$.

iii) $\delta l = \varepsilon \times l = 0.0004365 \times 4 = 0.00174 \ m$

$\delta l = \frac{Pl}{AE} =$ $\quad = 1.74 \ mm$

Problem

- A steel bar 150 mm wide and 12.5 mm thick and 1 m long carries a pull of 180 kN. Find the extension in length and the contraction in width and thickness when the load is applied. Take E= 2x10^5 N/mm^2 and 1/m = 1/3.5



$$\delta l = \frac{Pl}{AE} = \frac{180\times10^3 \times 1000}{(150\times12.5)\,2\times10^5} = \underline{\underline{0.48\ mm}}$$

$$\text{linear strain} = \frac{\delta l}{l} = \frac{0.48}{1000} = 0.00048$$

$$\text{Poisson's ratio } \mu = \frac{\text{lateral Strain}}{\text{linear Strain}}$$

$$\text{lateral strain} = \mu \times \text{linear strain} = \frac{1}{3.5} \times 0.00048 = 0.00013714$$

$$\text{lateral Strain} = \frac{\delta w}{w} = 0.00013714$$

$$\delta w = 0.00013714 \times 150 = \underline{\underline{0.0205\ mm}}$$

$$\text{lateral strain} = \frac{\delta t}{t} = 0.00013714$$

$$\delta t = 0.00013714 \times 12.5 = \underline{\underline{0.001714}}\ mm$$

Problem

- A bar of certain material 50 mm square is subjected to an axial pull of 150 kN. The extension over a length of 100 mm is 0.05 mm and decrease in each side is 0.00625 mm. calculate the Young's modulus, Poisson's ration, rigidity modulus and bulk modulus

Contact

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Thank you

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STRENGTH OF MATERIALS FOR MECHANICAL ENGINEERS (CE8395)

Unit-2

(TRANSVERSE LOADING ON BEAMS AND STRESSES IN BEAM)

by

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Course Objectives:

1. To understand the concepts of stress, strain, principal stresses and principal planes.
2. To study the concept of shearing force and bending moment due to external loads in determinate beams and their effect on stresses.
3. To determine stresses and deformation in circular shafts and helical spring due to torsion.
4. To compute slopes and deflections in determinate beams by various methods.
5. To study the stresses and deformations induced in thin and thick shells.

Course outcomes

Students will be able to

1. Understand the concepts of stress and strain in simple and compound bars, the importance of principal stresses and principal planes.
2. Understand the load transferring mechanism in beams and stress distribution due to shearing force and bending moment.
3. Apply basic equation of simple torsion in designing of shafts and helical spring
4. Calculate the slope and deflection in beams using different methods.
5. Analyze and design thin and thick shells for the applied internal and external pressures.

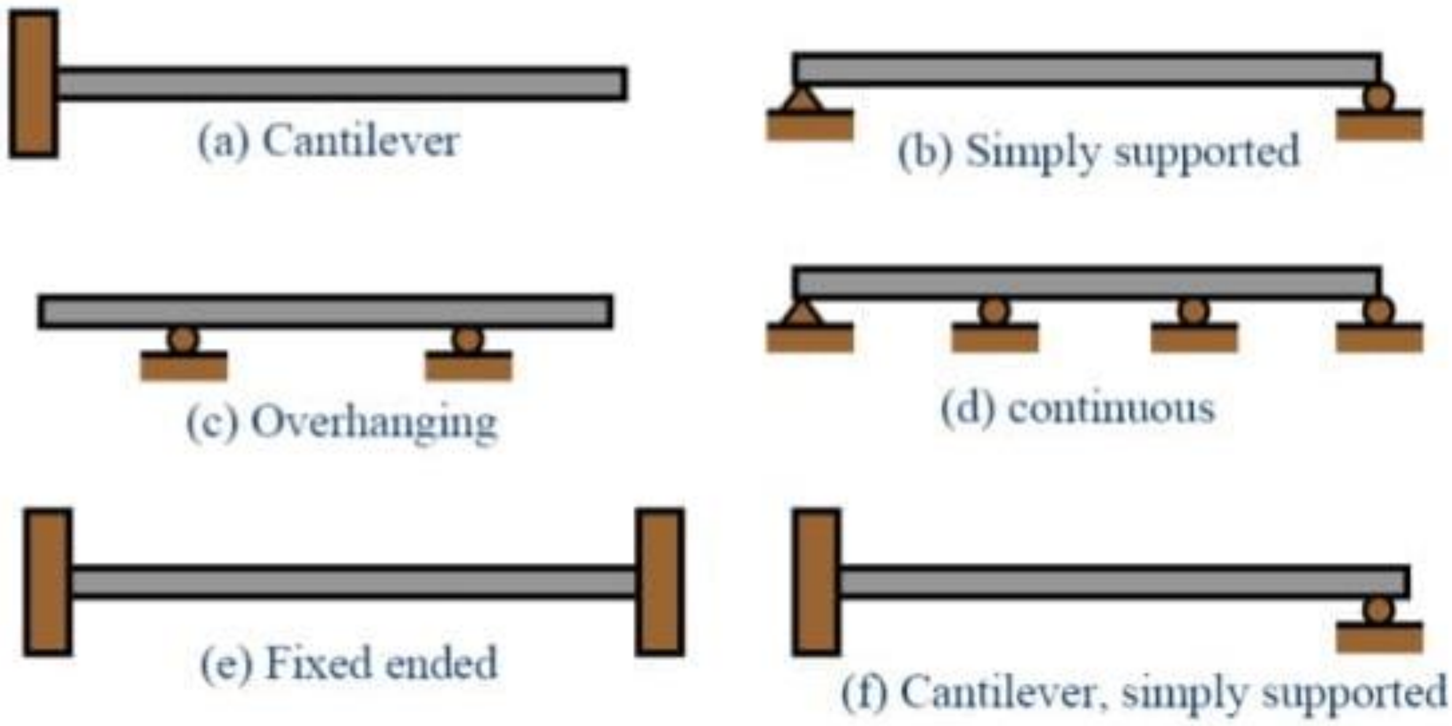
Syllabus Unit-2

UNIT II TRANSVERSE LOADING ON BEAMS AND STRESSES IN BEAM

Beams – types transverse loading on beams – Shear force and bending moment in beams – Cantilevers – Simply supported beams and over –hanging beams. Theory of simple bending– bending stress distribution – Load carrying capacity – Proportioning of sections – Flitched beams – Shear stress distribution.

Types of beams

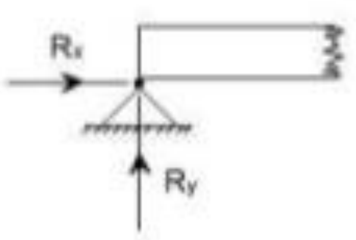
- Beams can be classified according to the manner in which they supported



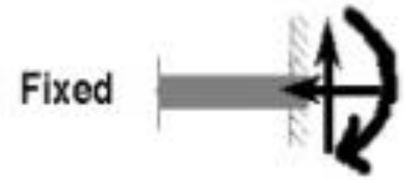
Types of supports

Roller support

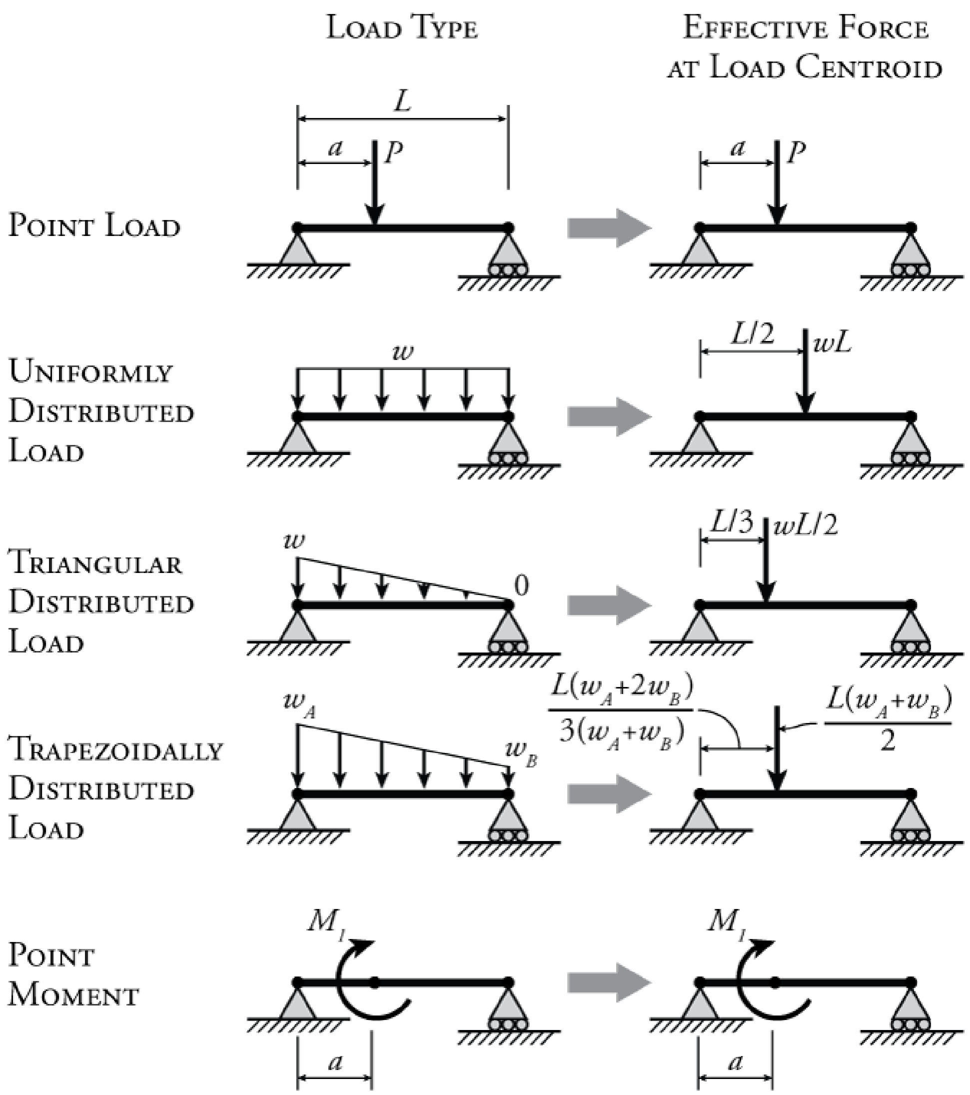
Hinged support



Fixed support



Types of loads

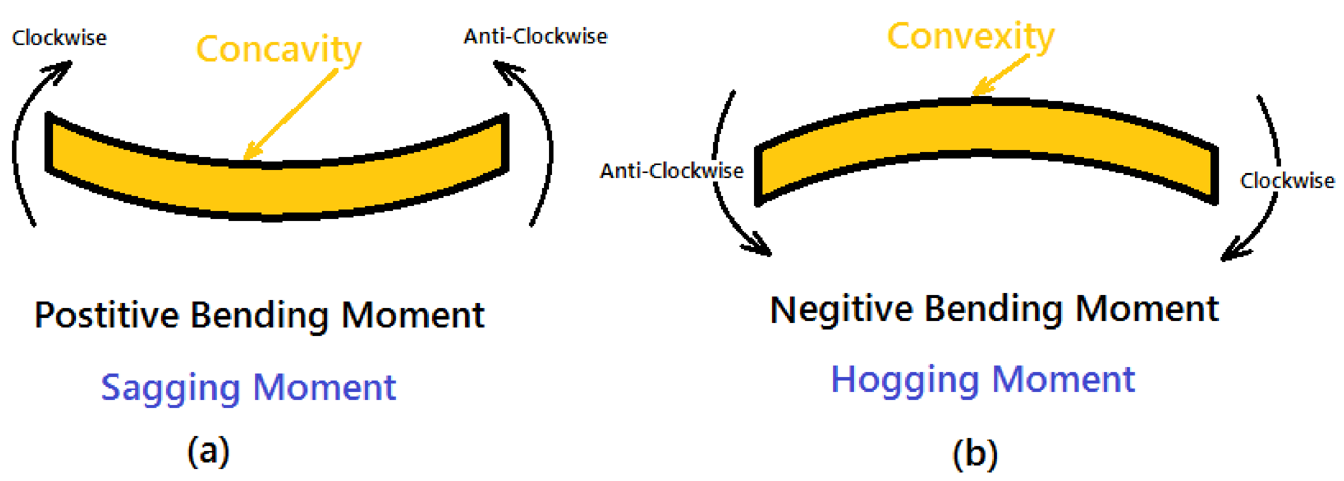


Shear force and Bending Moment

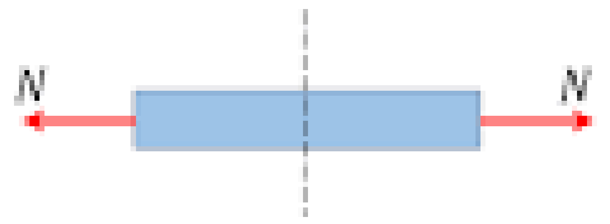
- Shear force
  - The shear force at any section is basically the algebraic sum of the lateral forces acting on either side of the section.
- Bending moment
  - The algebraic sum of the moments about the section of all forces acting on either side of the section

Bending moment – sagging and hogging

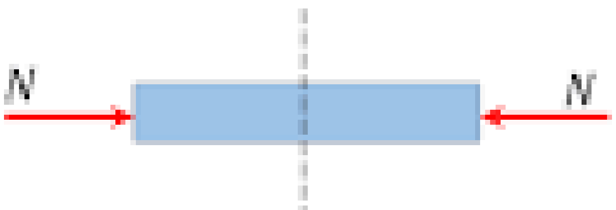
- A sagging moment will make the beam concave (positive) upwards at that section, and vice versa for a hogging moment (i.e. negative).



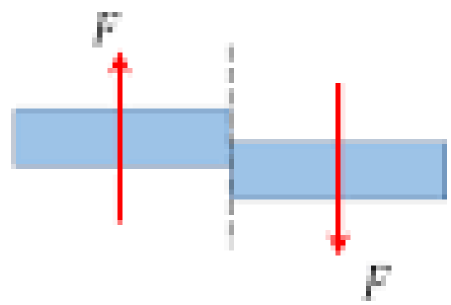
Sign conventions



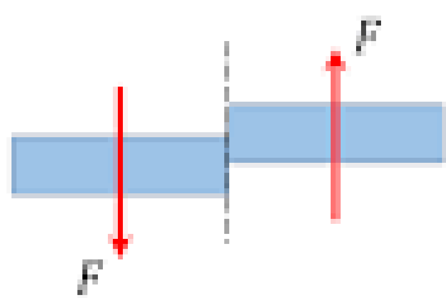
Positive axial force



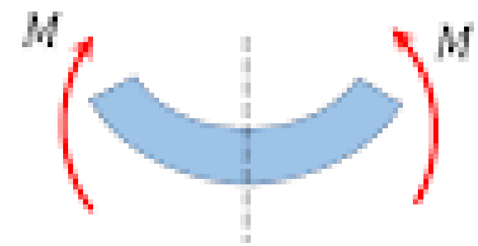
Negative axial force



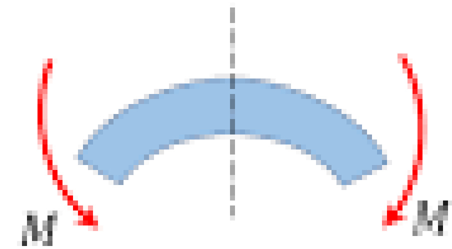
Negative shear force



Positive shear force



Positive bending moment



Negative bending moment

Relation between load intensity, shear force and bending moment

- The shear force is equal to the rate of change of bending moment with respect to x.

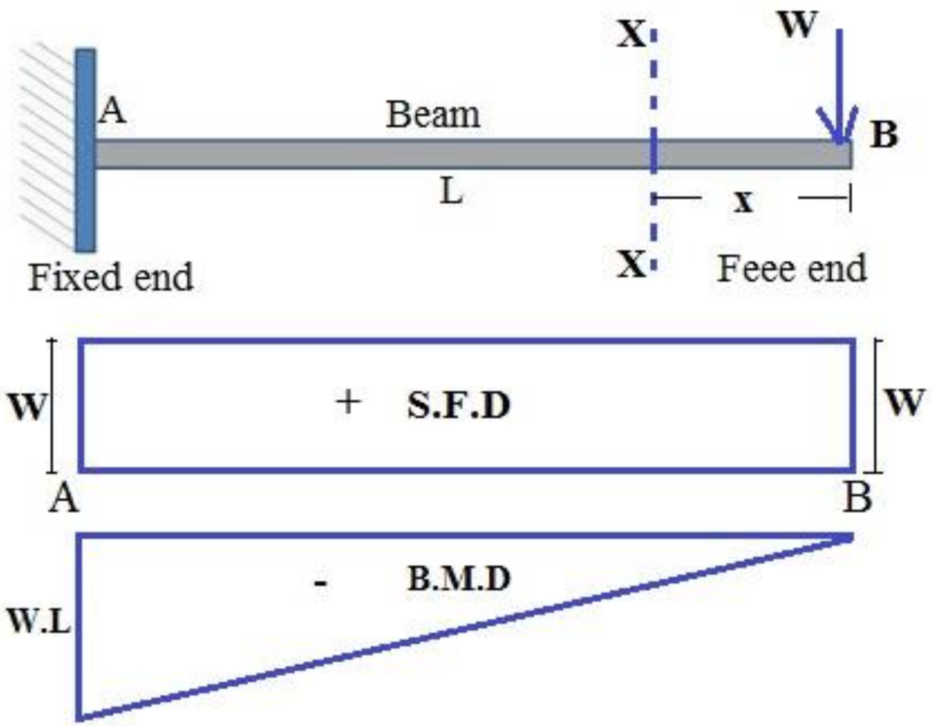
$$\mathbf{F = \frac{dM}{dx}}$$

- Slope of the moment curve = Shear Force
- The intensity of loading is equal to rate of change of shear force with respect to x.

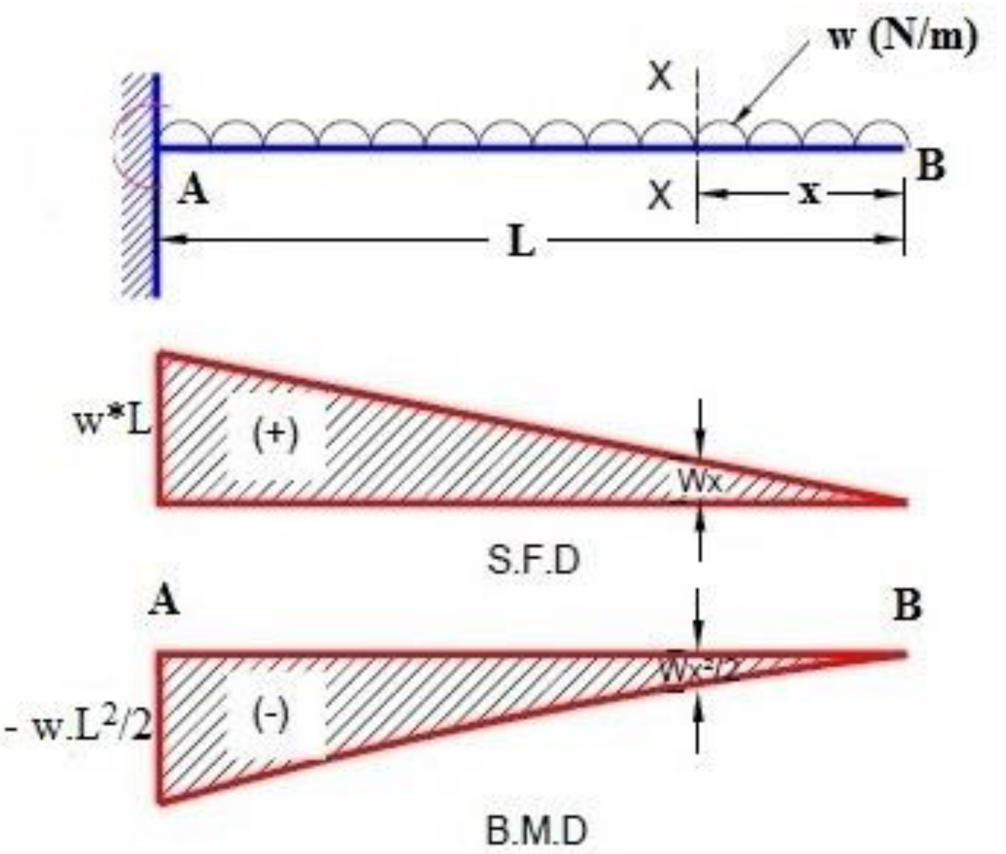
$$\mathbf{W = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}}$$

Slope of the shear diagram = - Value of applied loading

Cantilever with point load



Cantilever with udl



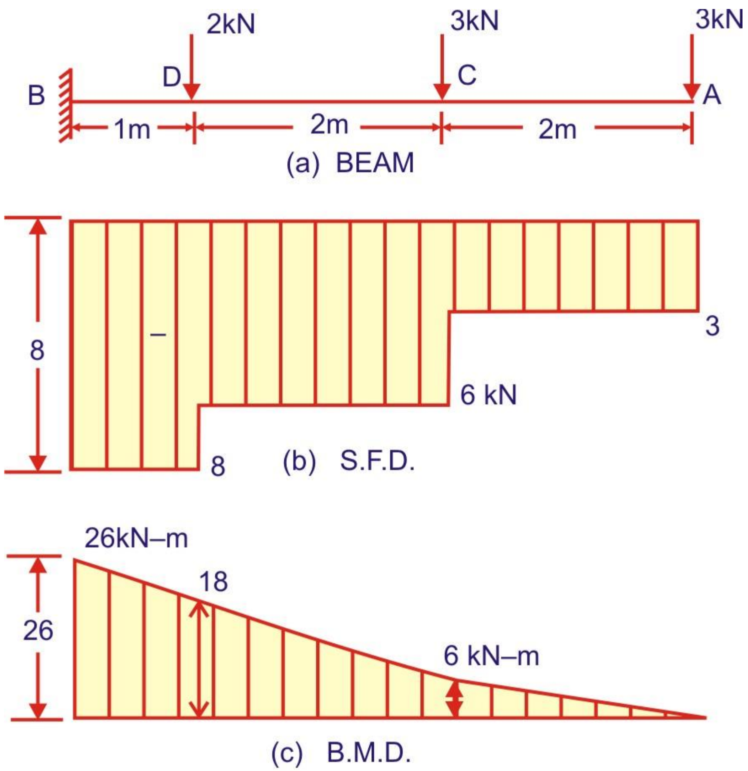

2kN
3kN
3kN
B
D
C
A
1m
2m
2m
(a) BEAM
8
–
3
6 kN
8
(b) S.F.D.
26kN–m
18
26
6 kN–m
(c) B.M.D.

Fig. 3.9.

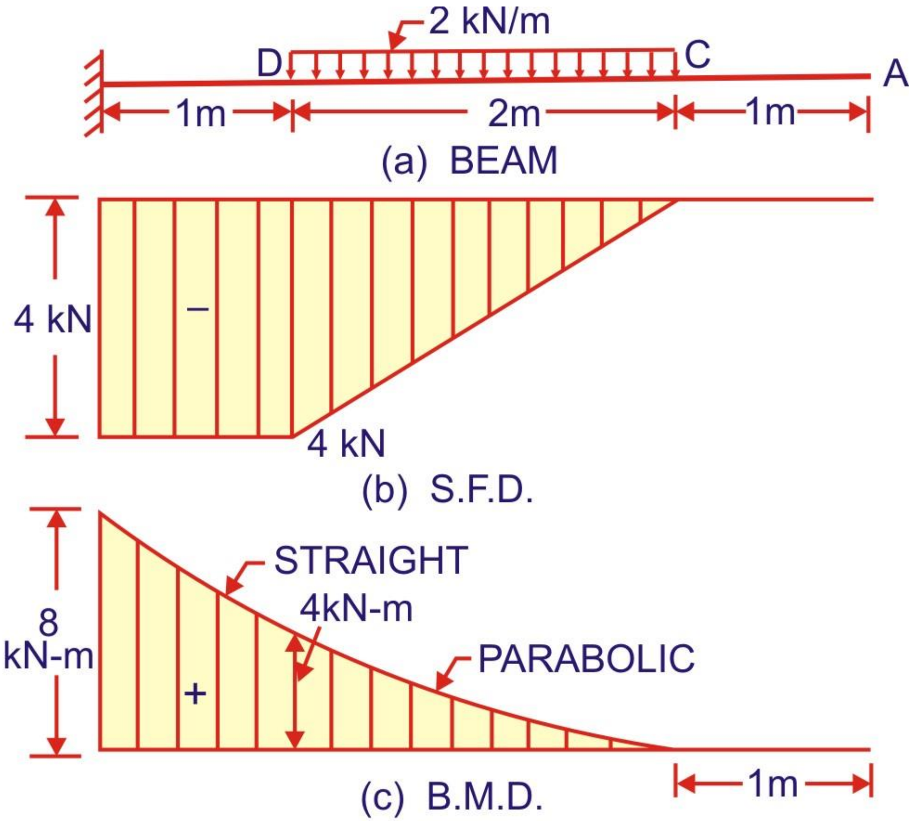

2 kN/m
D
C
A
1m
2m
1m
(a) BEAM
4 kN
–
4 kN
(b) S.F.D.
8
kN-m
STRAIGHT
4kN-m
PARABOLIC
+
1m
(c) B.M.D.

Fig. 3.13.

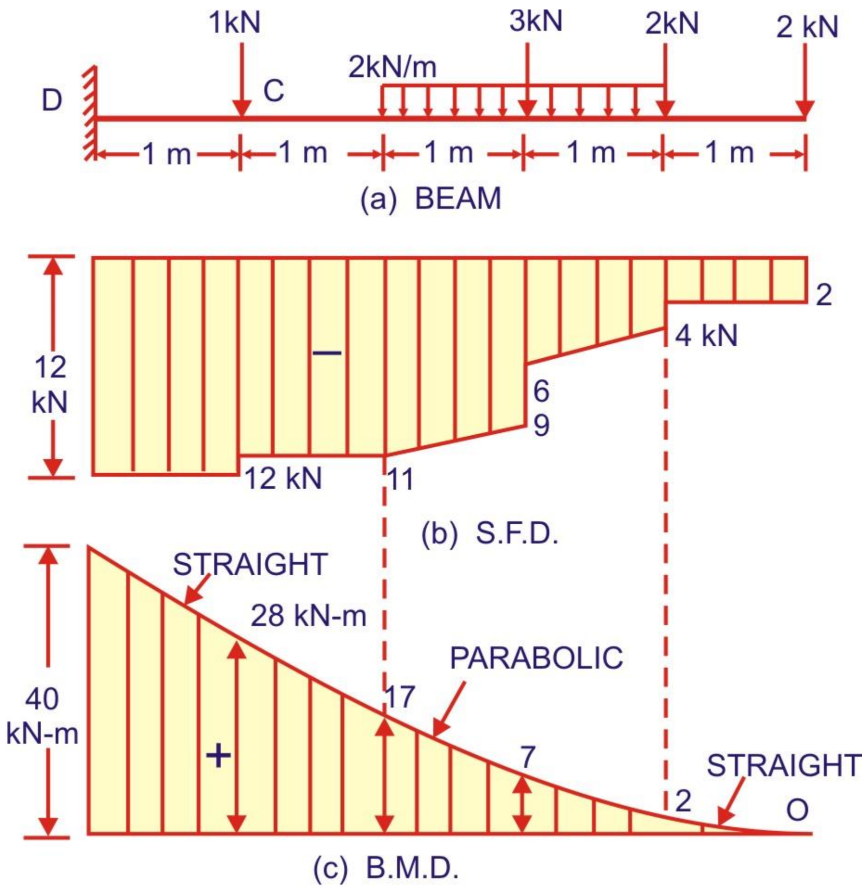
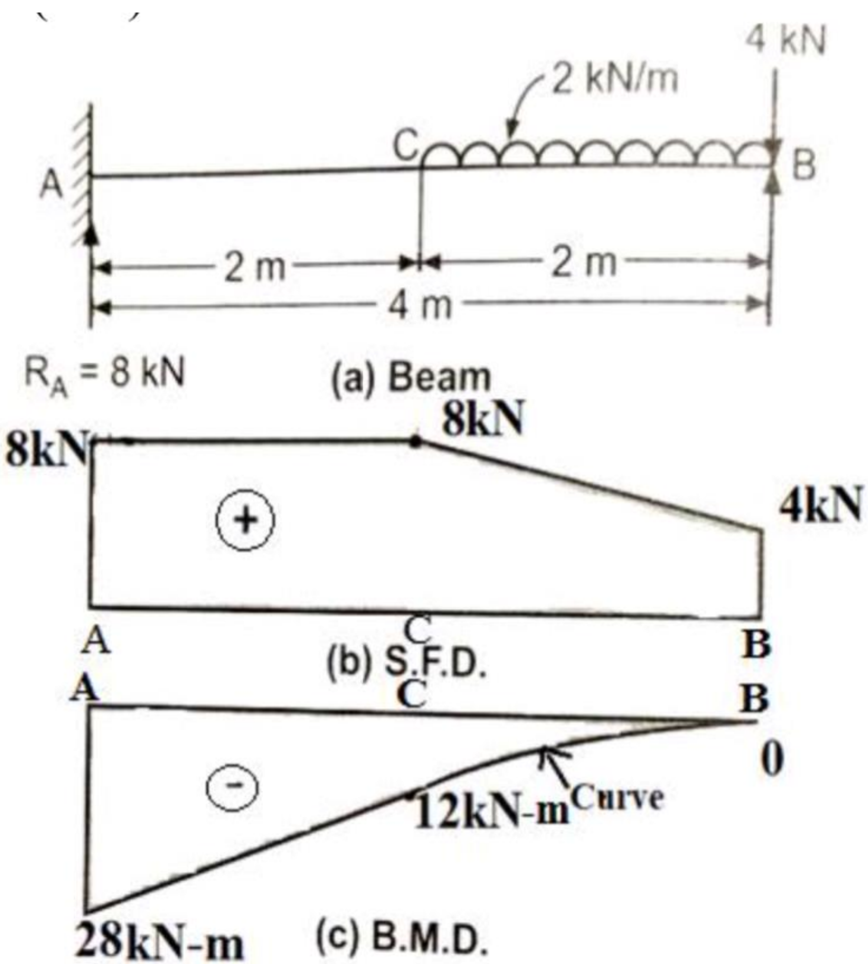
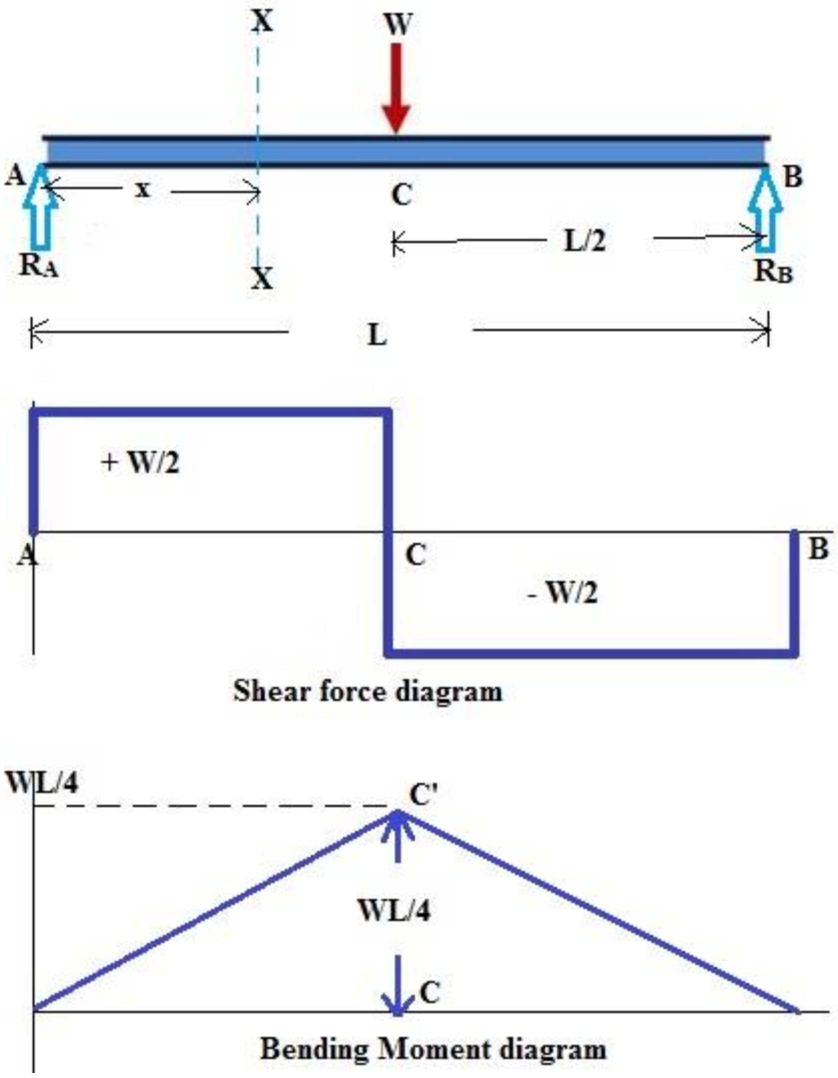

1kN
3kN
2kN
2 kN
2kN/m
D
C
1 m
1 m
1 m
1 m
1 m
(a) BEAM
12
kN
–
2
4 kN
6
9
12 kN
11
(b) S.F.D.
40
kN-m
STRAIGHT
28 kN-m
PARABOLIC
17
+
7
STRAIGHT
2
O
(c) B.M.D.

Fig. 3.14.


4 kN
2 kN/m
C
A
B
2 m
2 m
4 m
R_A = 8 kN
(a) Beam
8kN
8kN
4kN
A
C
B
(b) S.F.D.
A
C
B
0
12kN-m
Curve
28kN-m
(c) B.M.D.

SFD and BMD for Simply supported beams



SFD and BMD for Simply supported beams

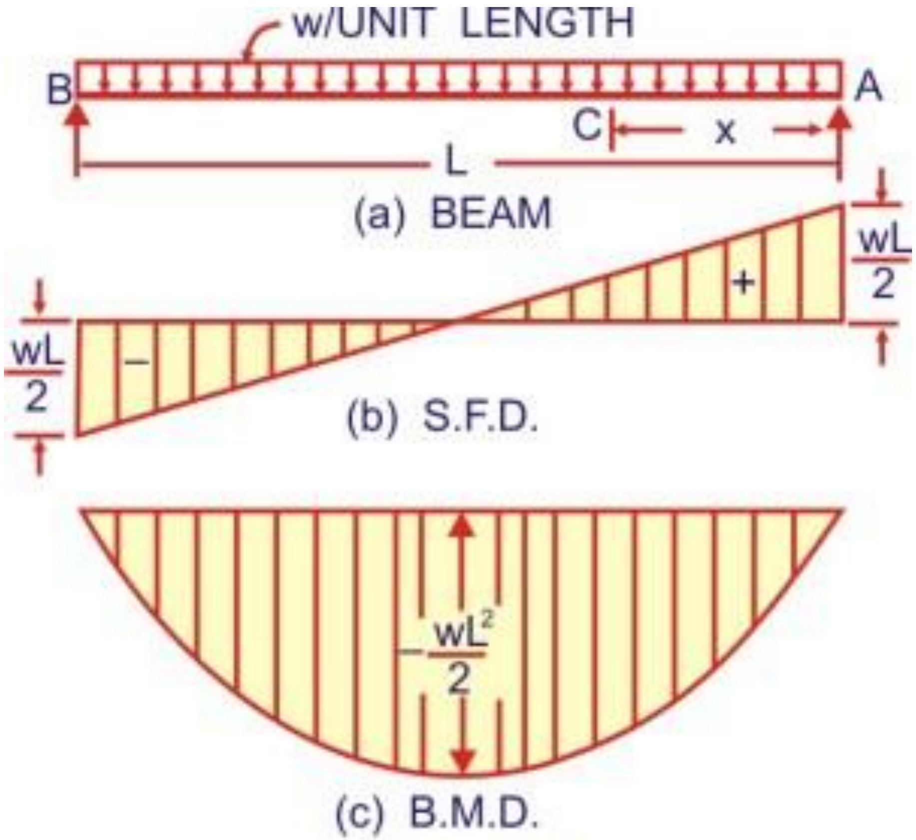
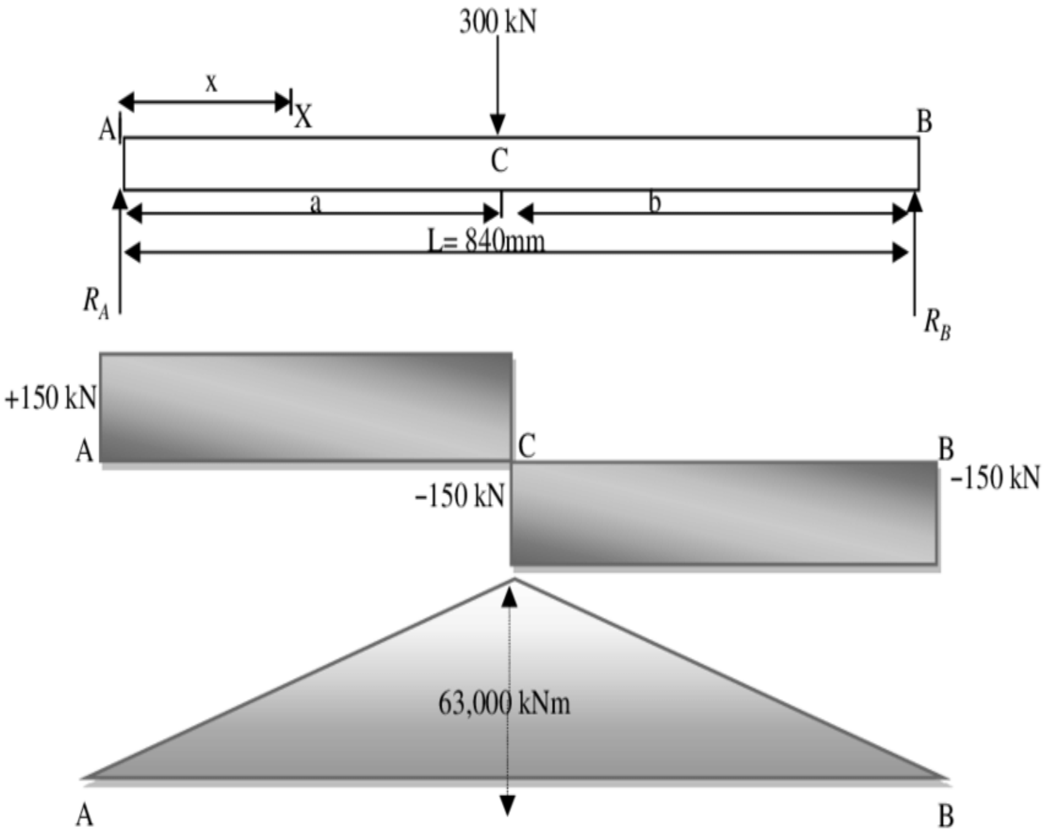


Fig. 3.18.

SFD and BMD for Simply supported beams



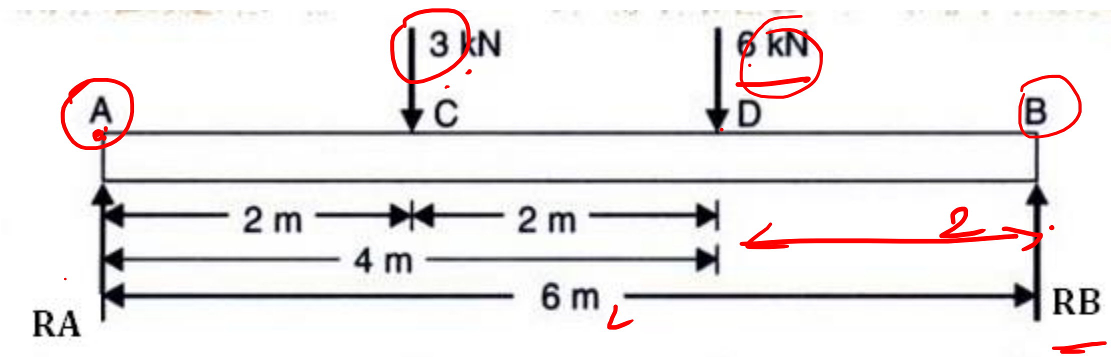
SFD and BMD for Simply supported beams

$\Sigma V = 0$

$R_A + R_B - 3 - 6 = 0$

$R_A + R_B = 9$ —— (1)

$\Sigma M = 0$ $\Sigma M_A = 0$.

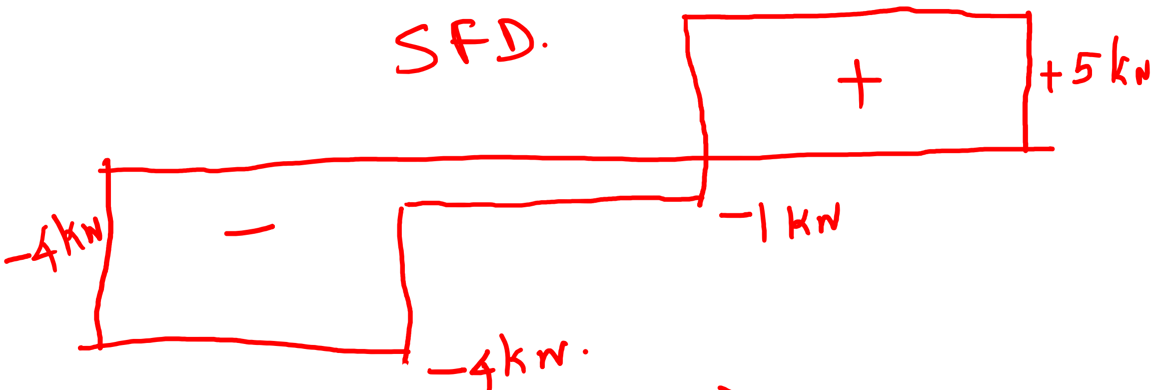


$R_B \times 6 - 6 \times 4 - 3 \times 2 = 0$.

$R_B \times 6 = 30$

$R_B = 5\ kN$

$R_A = 4\ kN$.

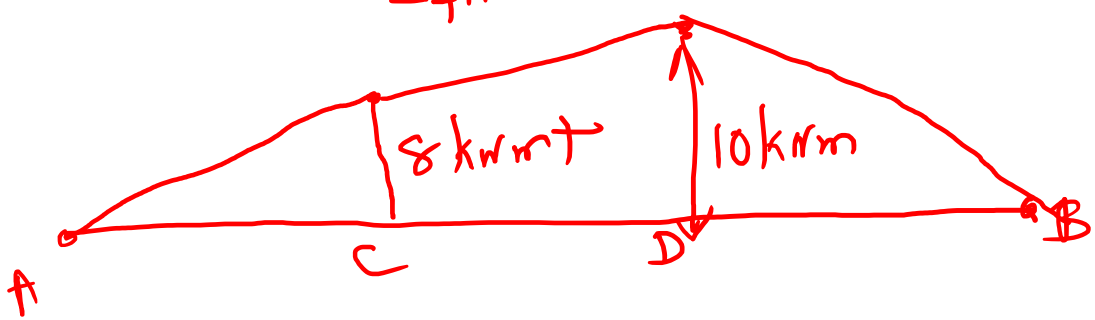


Bending moment

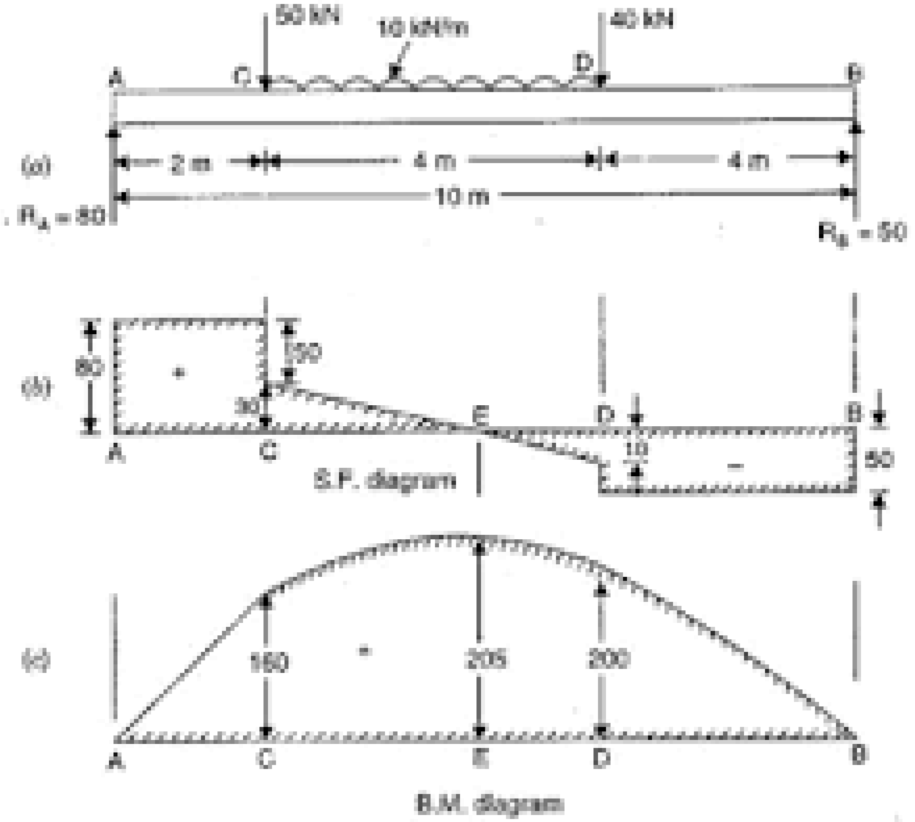
$M_D = 5 \times 2 = 10\ kNm$

$M_C = 5 \times 4 - 6 \times 2$

$= 20 - 12 = 8\ kNm$.

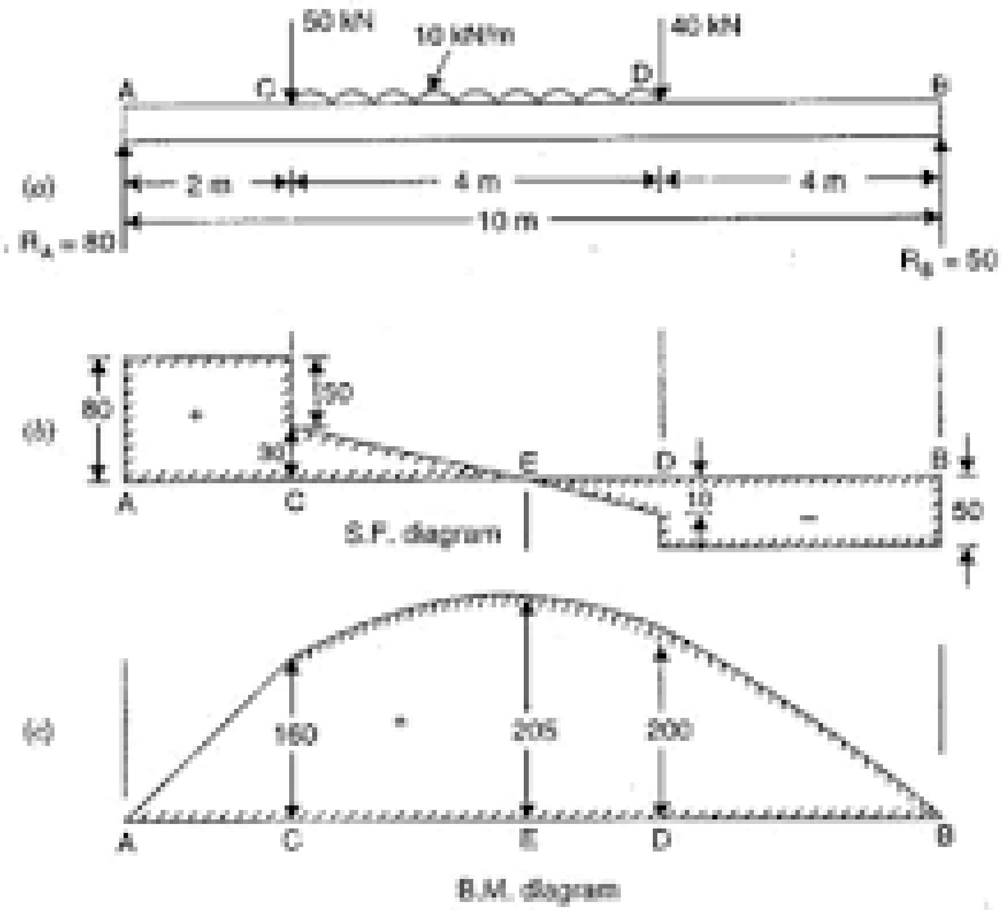


SFD and BMD for Simply supported beams



SFD and BMD for Simply supported beams

The maximum bending moment occurs in a beam, when the **shear force** at that section is zero or changes the sign



SF at Section X

$F_x = 50 - 40 - 10(x-4) = 0$

$10 - 10(x-4) = 0$

$10(x-4) = 10$

$x - 4 = 1$

$x = 5\text{ m}$

BM at Section X

$M_x = 50x - 40(x-4) - \frac{10(x-4)^2}{2}$

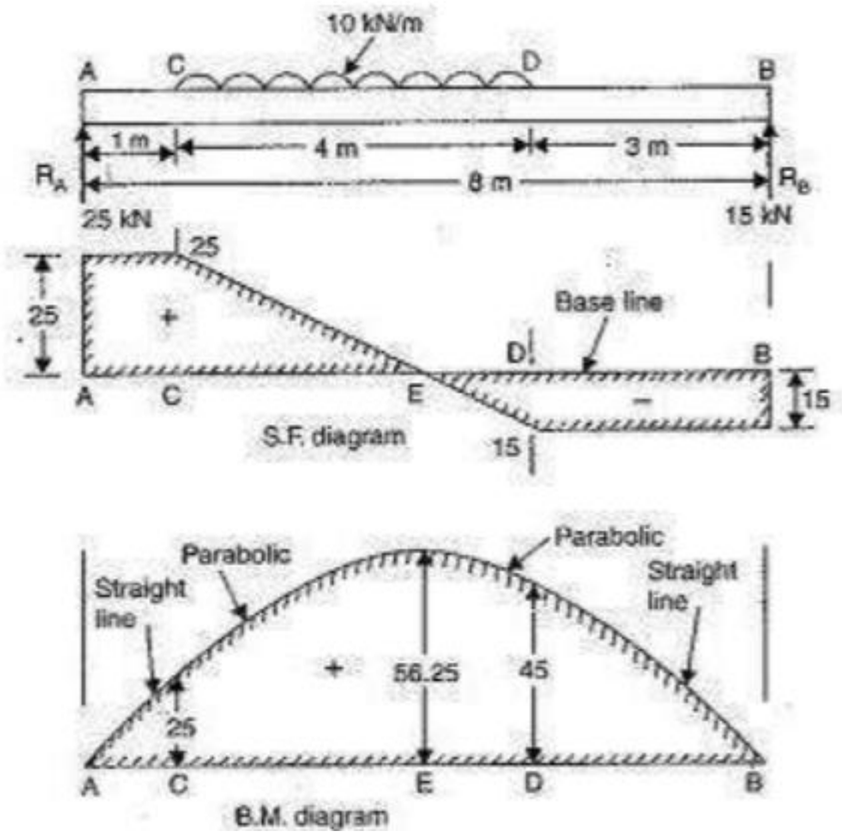
$= 50 \times 5 - 40(5-4) - 5(5-4)^2$

$= 250 - 40 - 5$

$= 205\text{ kNm}$

Maximum Bending Moment

Draw the Shear Force Diagram and Bending Moment Diagram for a simply supported beam of length 8m and carrying a uniformly distributed load of 10 kN/m for a distance of 4m as shown in figure.



SF at Section X

$F_x = 15 - 10(x-3) = 0.$

$10(x-3) = 15$

$x - 3 = 1.5$

$x = 4.5\text{m}$ from end B

Max. BM

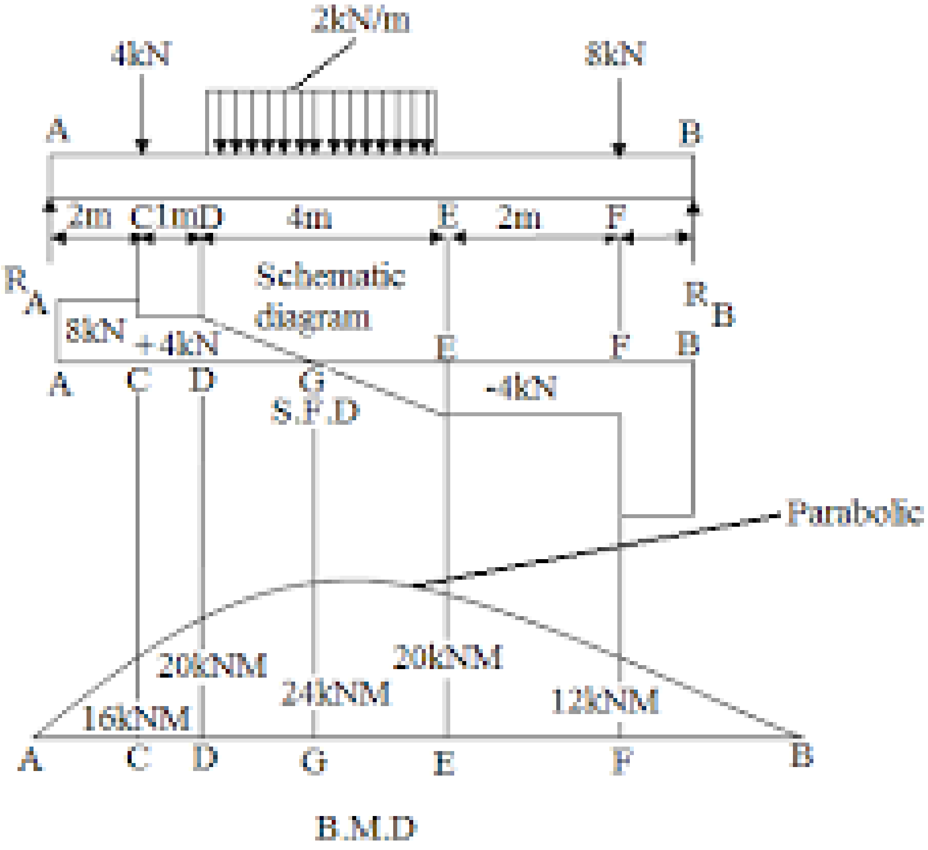
$M_x = 15x - 10(x-3)\frac{(x-3)}{2}$

$= 15x - 5(x-3)^2$

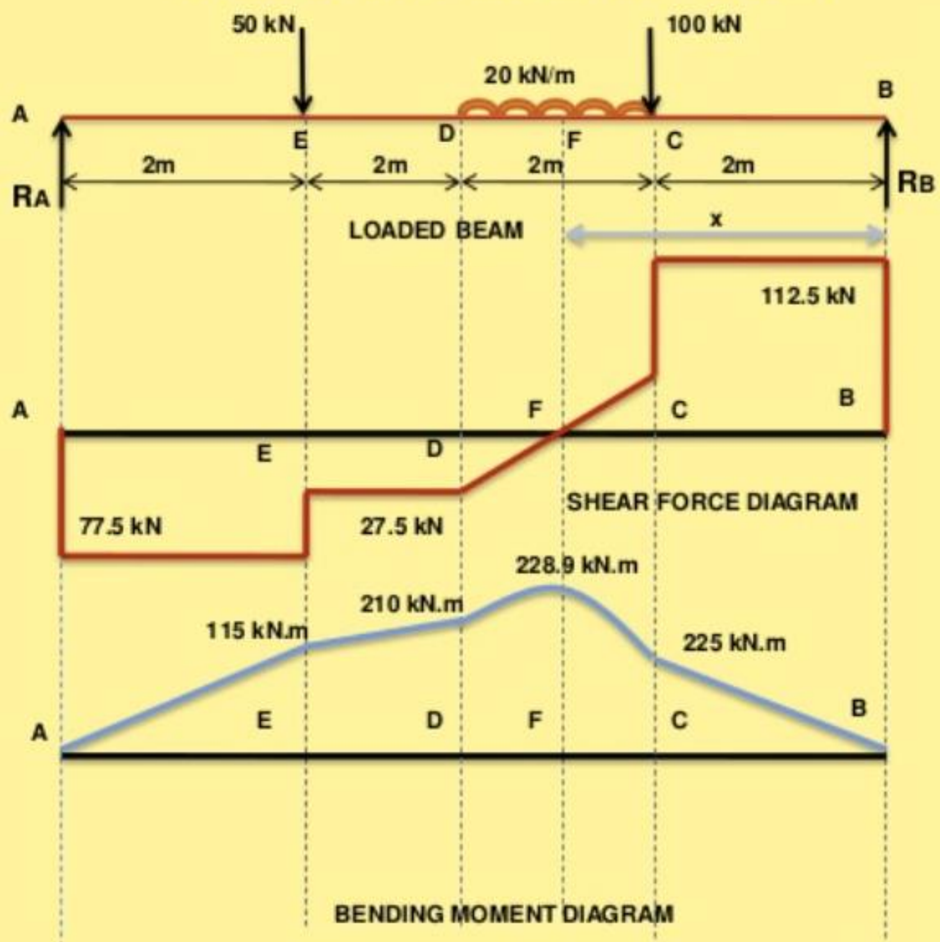
$= 15(4.5) - 5(4.5-3)^2$

$= 56.25$ kNm

SFD and BMD for Simply supported beams



EFFECT OF UNIFORMLY DISTRIBUTED LOAD



Calculations

Reactions

$R_A + R_B = 100 = 20 \times 2 + 50 = 190$

M_A,

$R_B \times 8 = 100 \times 6 + 20 \times 2 \times 50 \times 2$

$R_B = 112.5$ kN, RA = 77.5 kN

Shear Force

S.F at B = 112.5 kN (+ ve)

S.F at C = 112.5 - 100 = 12.5 kN

S.F at D = 112.5 - 100 – 20 x 2 = -27.5 kN

S.F at E = 112.5 – 100 – 20 X 2 – 50 = -77.5 kN

S.F. at A = – 77.5 kN

S.F. at F = 0

S.F. at F = 112.5 – 100 – 20 (x – 2)

X = 2.625 m

Bending Moment

BM_A = 0 , **BM_B = 0**

BM_C = 112.5 X 2 = 225 kN.

BM_D = 112.5 X 4 – 100 X 2 – 20 X 2 X 2/2
= 210 kN.m

BM_E = 112.5 X 6 – 100 X 4 – 20 X 2 X 3
= 155 kN.m

BM_F = 112.5 X 2.625 – 100 X 0.625 – 20 X 0.625 X 0.625/2 = 228.9 kN.m

Over hanging beam

- Draw the SFD and BMD for the overhanging beam carrying UDL of 2 kN/m over the entire length and a point load of 2 kN as shown in Fig. Also locate the point of contraflexure

$$M_x = -2x + 12(x-2) - x^2 = 0$$

$$x^2 + 2x - 12(x-2) = 0$$

$$x^2 + 2x - 12x + 24 = 0$$

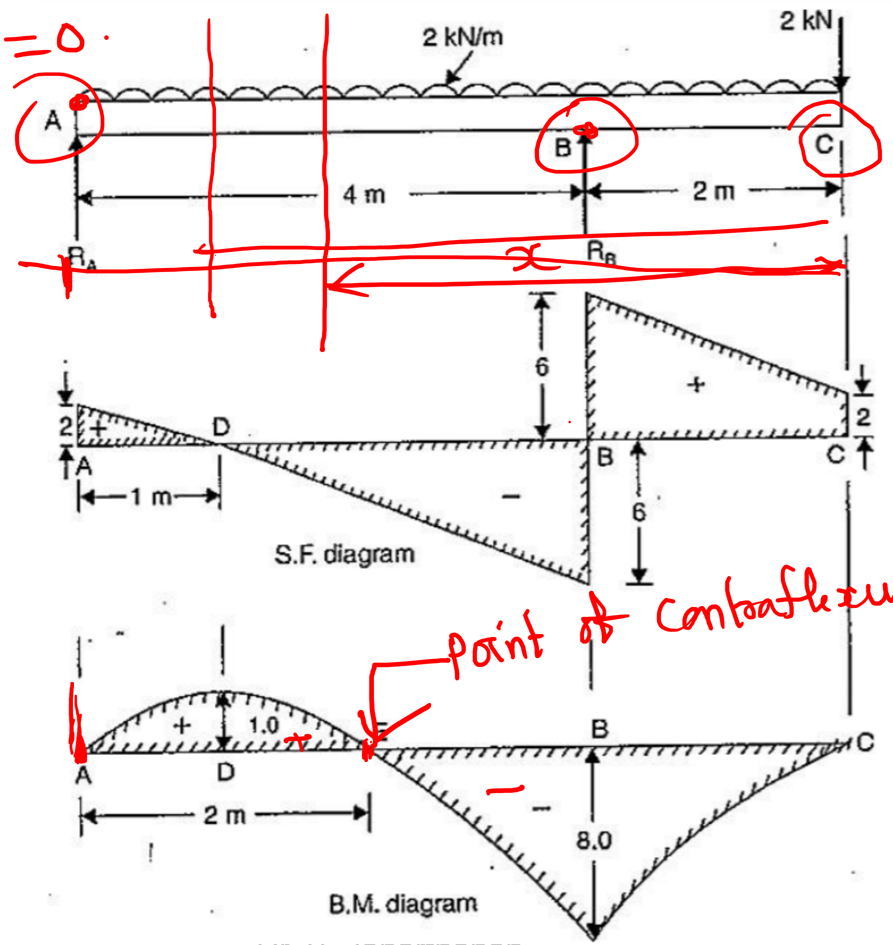
$$x^2 - 10x + 24 = 0$$

$$x^2 - 4x - 6x + 24 = 0$$

$$x(x-4) - 6(x-4) = 0$$

$$(x-4)(x-6) = 0$$

$$x = 4 \text{ or } 6$$



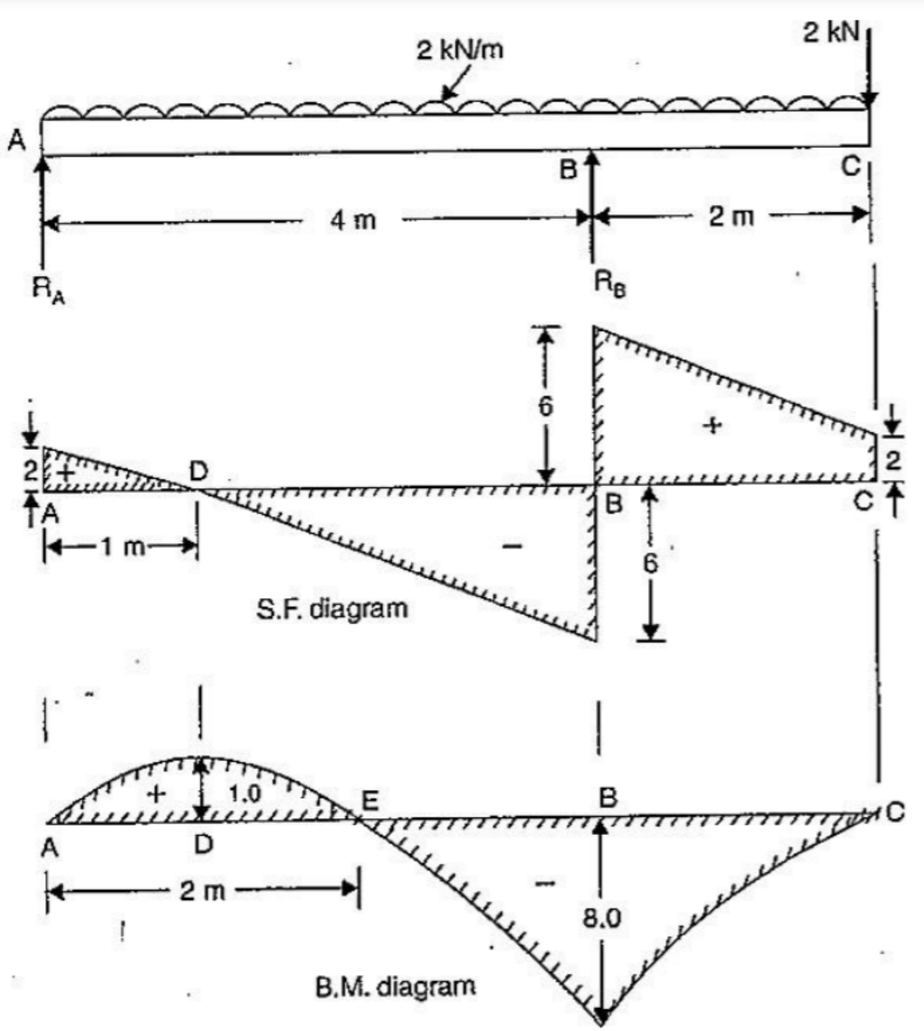
$x = 4$

Point Contraflexure.

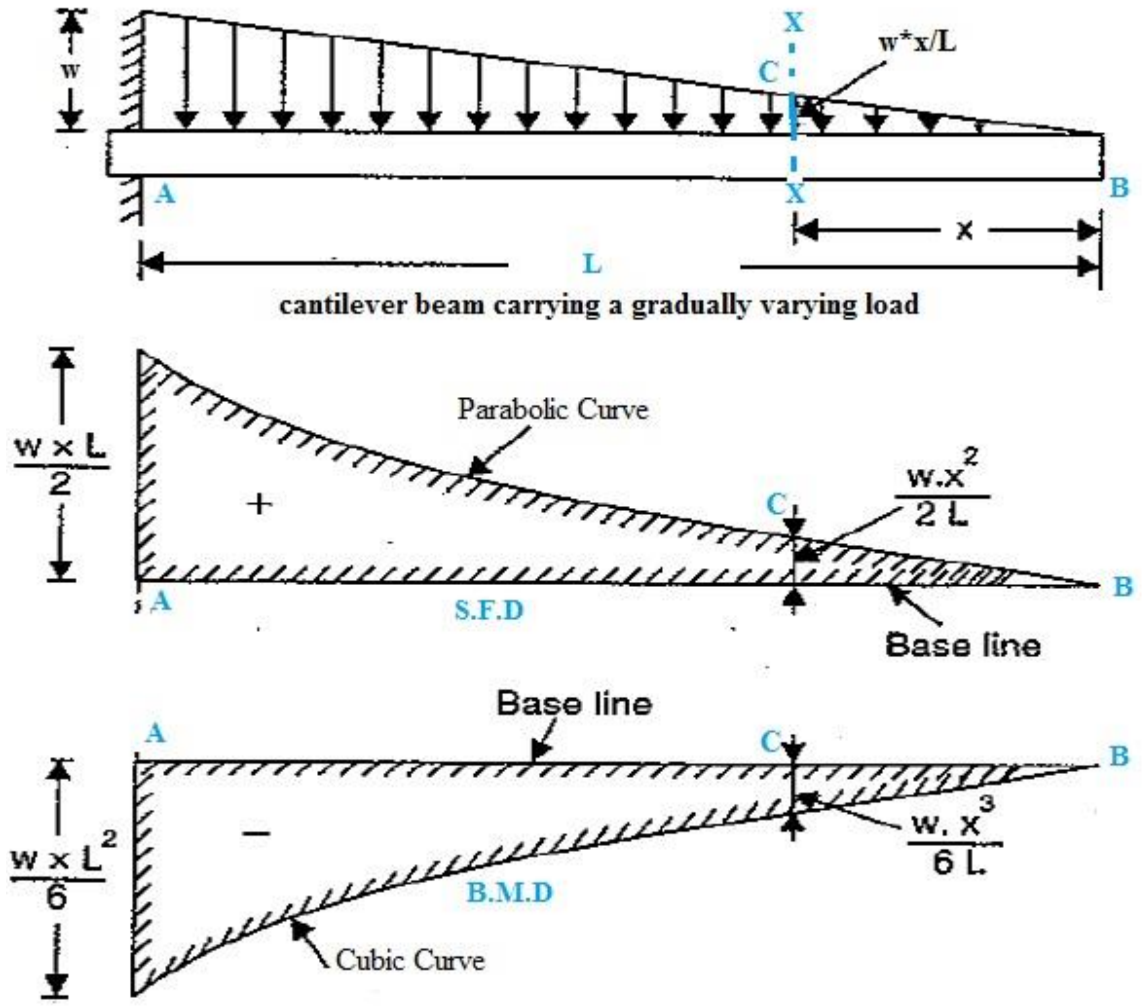
Point of contraflexure

- **POINT OF CONTRAFLEXURE**
- It **is** the **point** on a **beam** where bending moment changes its sign and its value **is** zero. This **point is** generally found in over hanging **beam**.

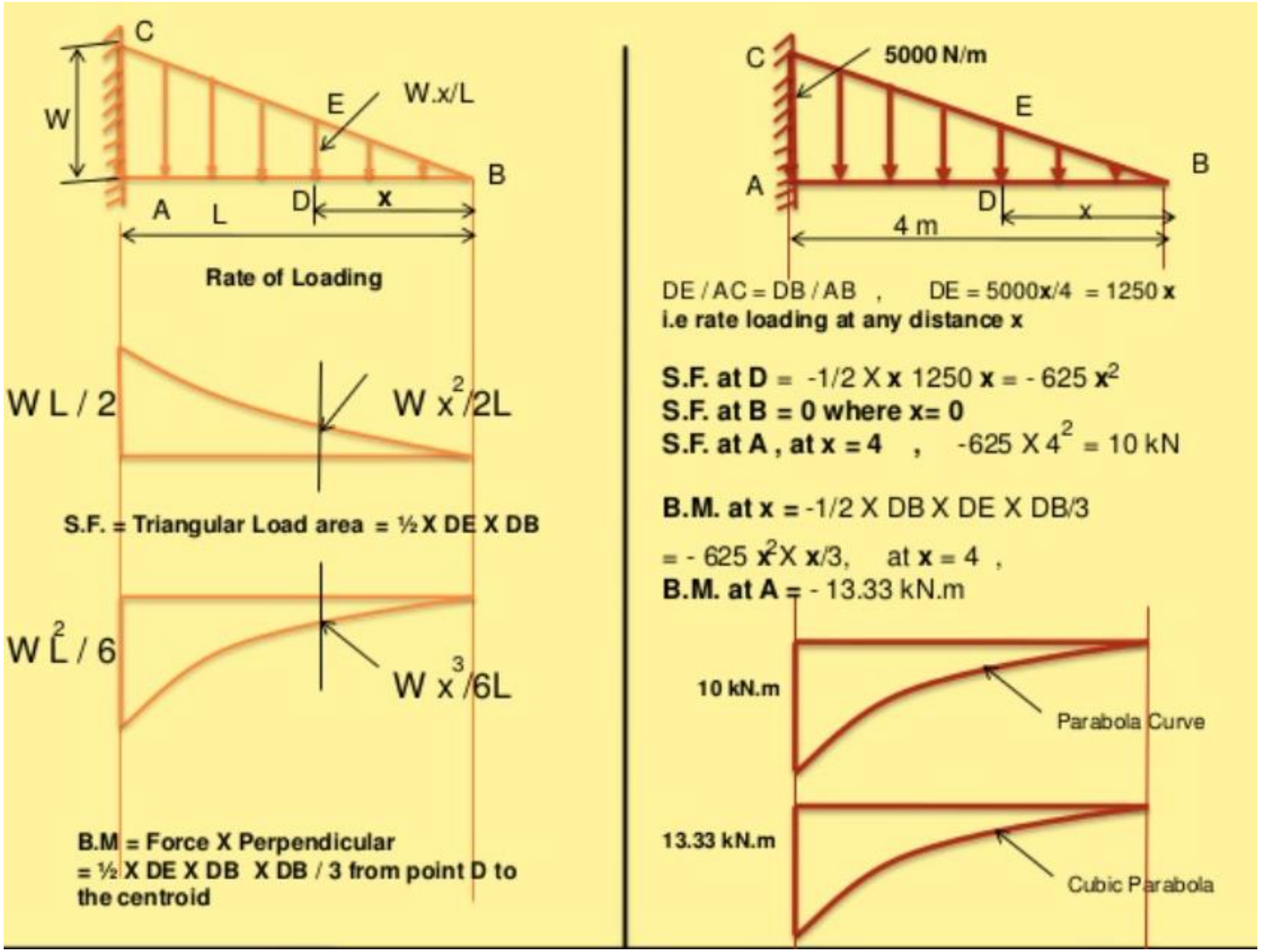
Overhanging beams



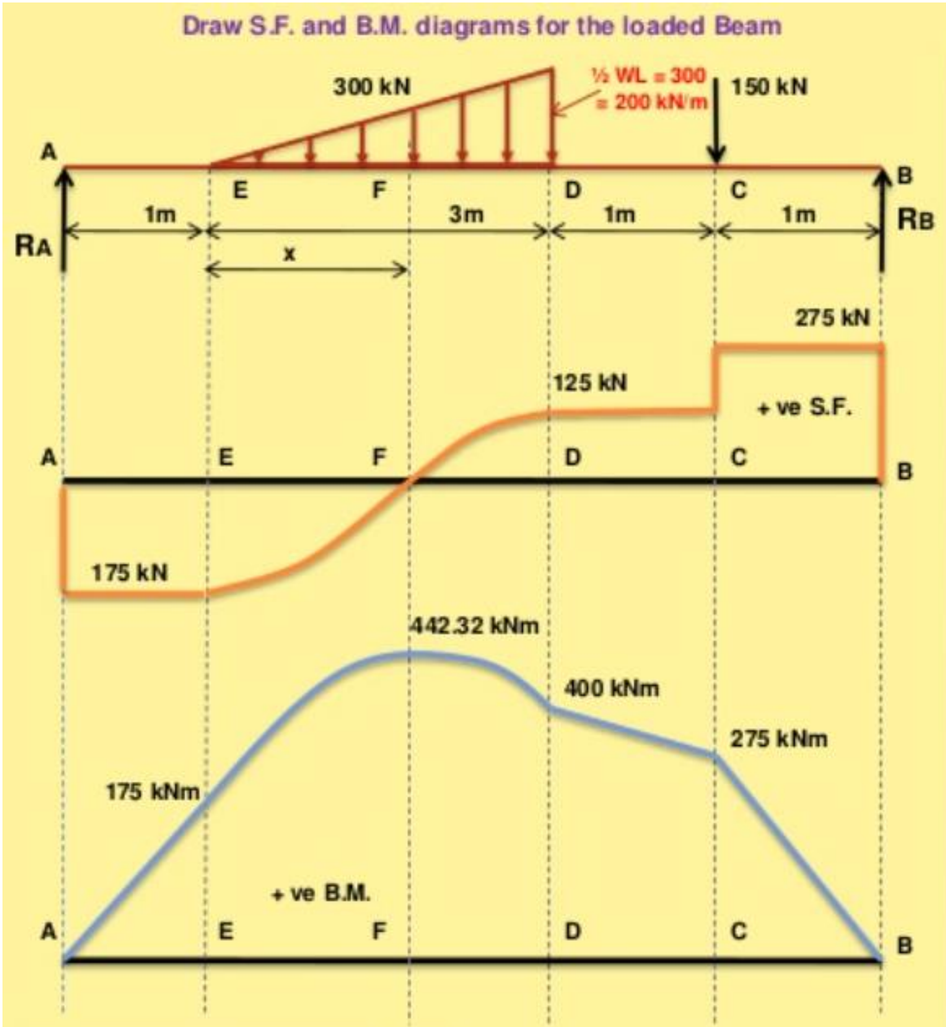
Uniformly varying load



Uniformly varying load



Uniformly varying load



Calculations

Reactions

$R_A + R_B = 150 + 300$

M_A,

$R_B \times 6 = 150 \times 5 + 300\,(2/3 \times 3 + 1)$

$R_B = 275$ kN, $R_A = 175$ kN

Shear Force

S.F at B = 275 kN

S.F at C = 275 – 150 = 125 kN

S.F at D = 125 kN

S.F at E = 275 – 150 – 300 = -175 kN

S.F at A = - 175 kN

Rate of Loading at distance x

$w = Wx/L =$ $w = 200\,x / 3$

S.F at F = $-175 + ½\ 200x / 3 \times x$

x= 2.29 m

Bending Moment

$BM_A = 0$, $BM_B = 0$,

$BM_C = 275 \times 1 = 275$ kN.m

$BM_D = 275 \times 2 - 150 \times 1 = 400$ kN.m

$BM_E = 175 \times 1 = 175$ kN.m ½ WL X L/3

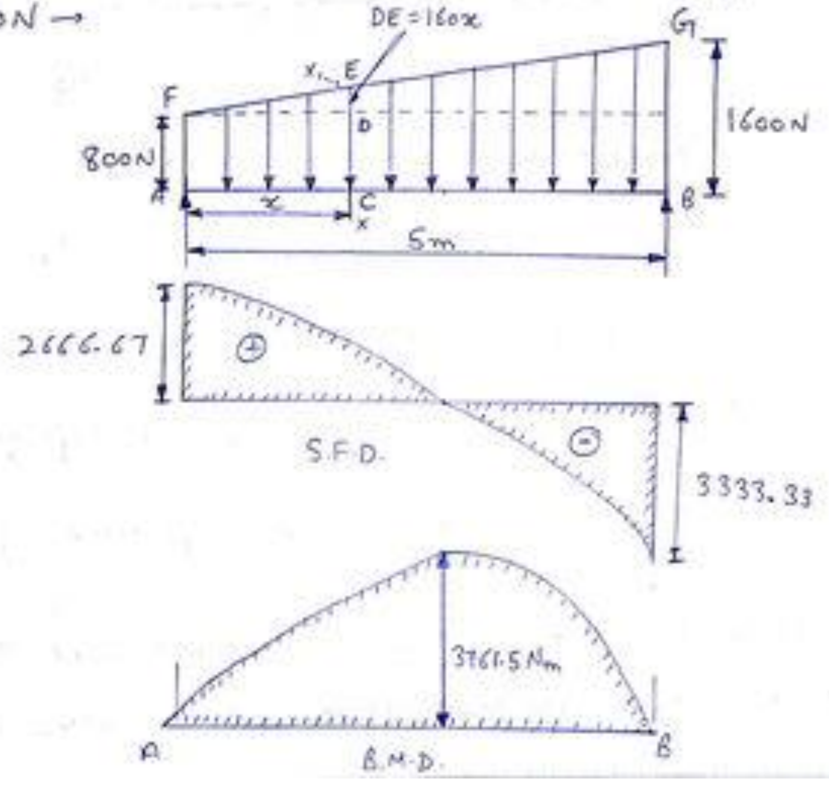
$BM_F = 175 \times 3.29 - (200 \times 2.29/3)\,(2.29/2 \times 2.29/3)$

$= 442.32$ kN.m

Uniformly varying load

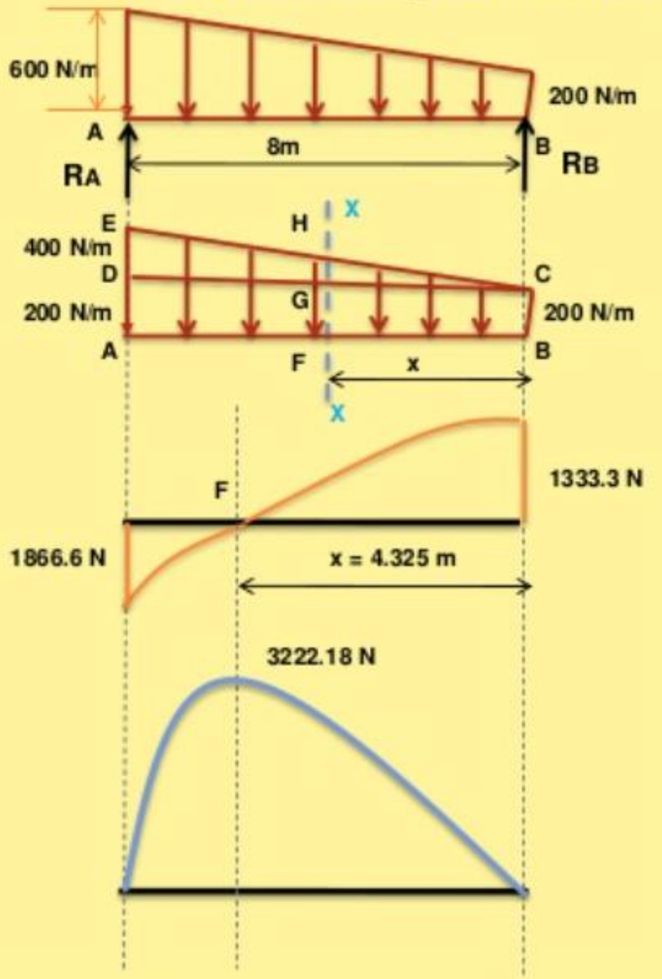
Problem → A Simply Supported Beam of length 5m carries a UVL 800 N/m run at one end to 1600 N/m run at other end. Draw S.F.D. & B.M.D. Also calculate position & Magnitude of max B.M.

SOLUTION →



Uniformly varying load

Draw S.F. and B.M. diagrams for the loaded Beam



Calculations

Reactions

M_A,

R_B X8= 200 X 8 X 4 + ½ X 400 X 8 X 8/3

R_B = 1333.33 N

R_A + R_B = 200 X 8 + ½ X 400 X 8

R_A = 1866.67 N

Rate of Loading at X-X = GH + GF

Rate of Loading at GH

DE/CD = GH/CG, GH = 400**x** /8 = 50**x**

Rate of Loading at GF = 200

Rate of Loading at X-X = GH + GF = 200 + 50**x**

Shear Force at P = 0

S.F. at F =1333.33 – (load BCGF + Load CGH

= 1333.33 – (200**x** + ½ X 50 **x** X **x**)

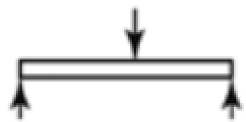
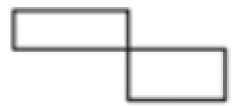
x = 4.326 (quadratic equation +ve value)

Bending Moment

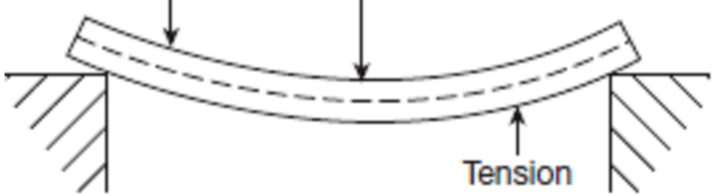
BM$_F$ = 1333.33x – 200x X x/2 – ½ X 50x X x X x/3…….(GH = 50x)

We have x = 4.326

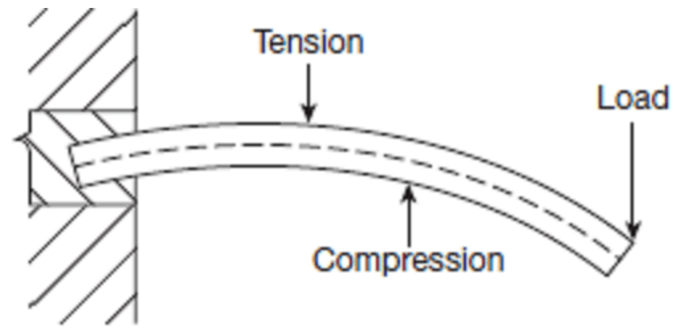
Max. B.M at F = 3436.14 N/m

Load	Slope for shear force	Slope for bending Moment
P 	Constant 	Linear 
Uniformly distributed load 	Linear 	Parabolic 
Uniformly varying load 	Parabolic 	Cubic 

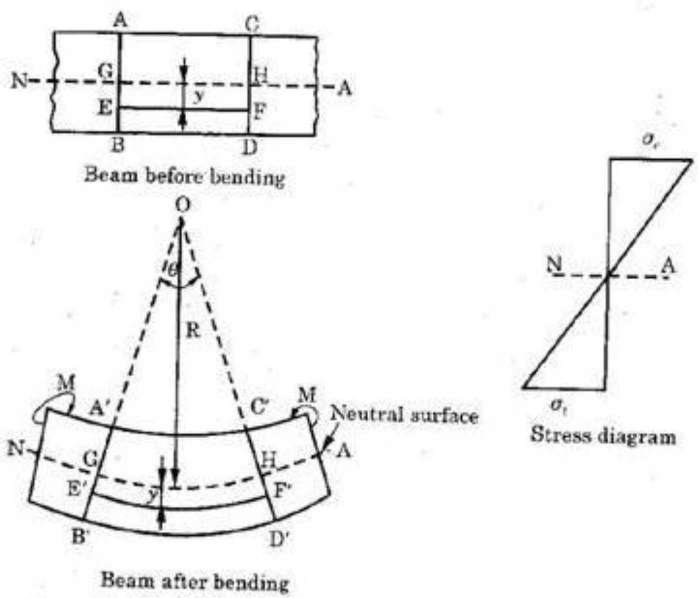
Theory of simple bending



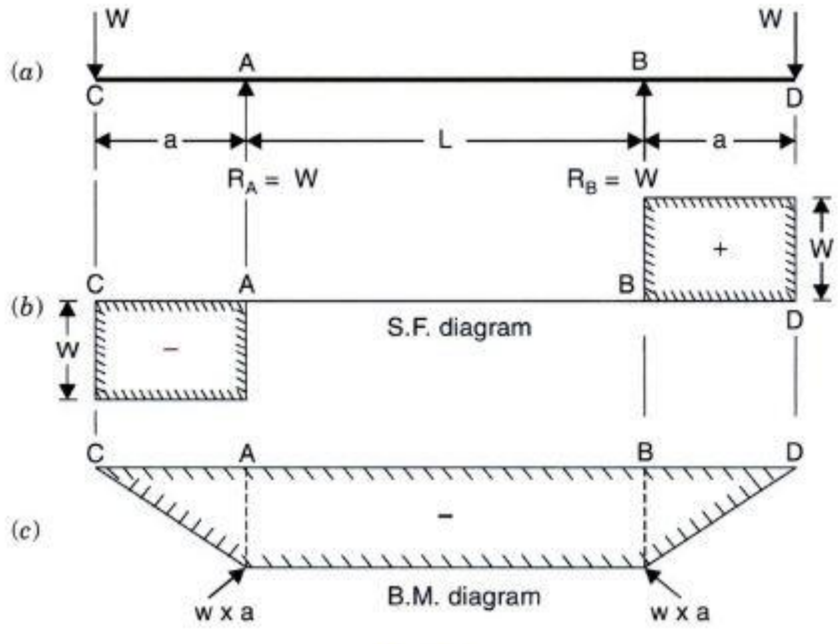
(a) Simply supported beam



(b) Cantilever beam



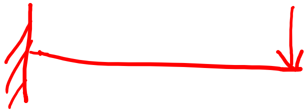
Pure bending or simple bending

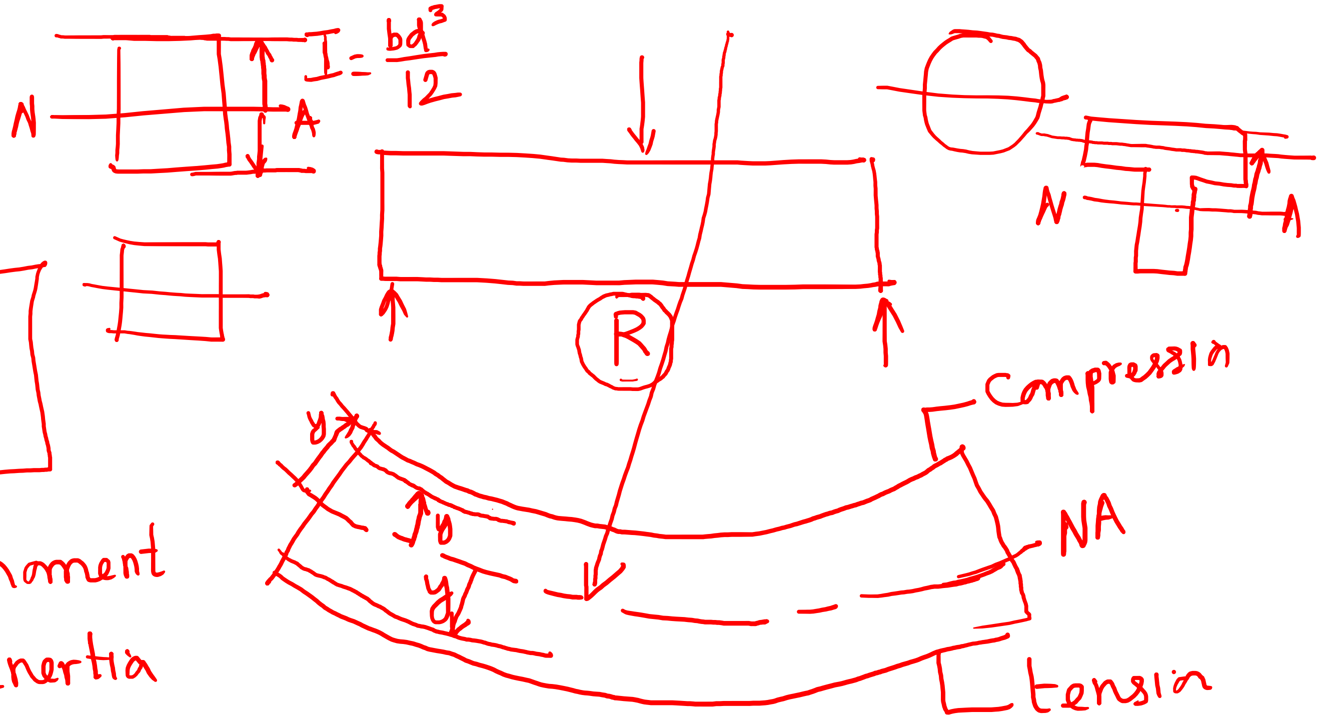


ASSUMPTIONS IN SIMPLE BENDING THEORY

- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions.
- Young's Modulus is the same in tension as in compression.
- Sections are symmetrical about the plane of bending.
- Sections which are plane before bending remain plane after bending.

- Bending stress
  - The stresses produced at the section to resist bending moment
- Shear stress
  - The stresses produced at the section to resist the shear force





Bending eqn

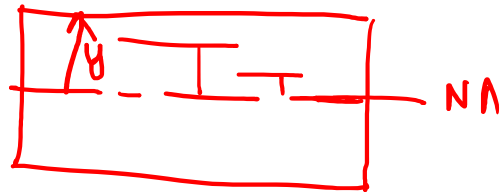
$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

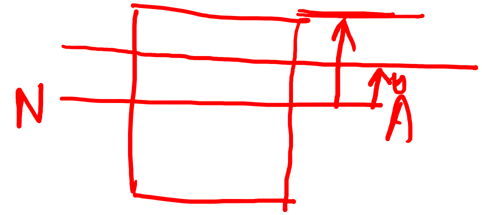
- ✓ M – max. Bending moment
- ✓ I – Moment of Inertia
- ✓ σ – Bending stress
- ✓ y – distance of fibre from Neutral Axis.
- ✓ E – Young's modulus
- ✓ R – Radius of curvature of N.A

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

Bending equation

$$\left(\frac{M}{I} = \frac{\sigma}{y}\right) = \frac{E}{R}.$$





$$\sigma_{max} = \frac{M}{I}\, y_{max}$$

$$= \frac{M}{\left(I/y_{max}\right)}$$

$$= \frac{M}{Z}$$

Define section modulus.

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

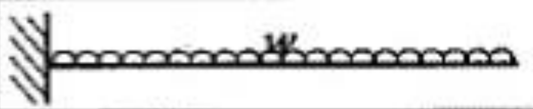
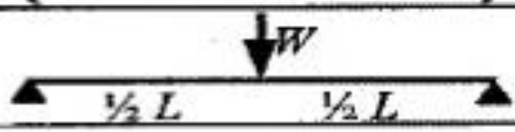
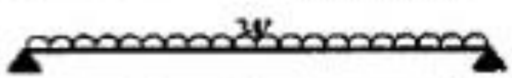
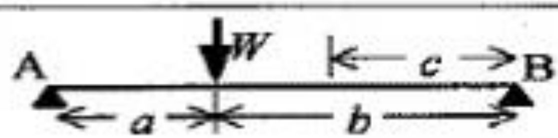
$$\frac{M}{I} = \frac{E}{R}$$

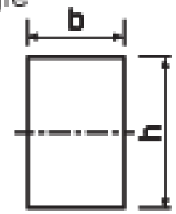
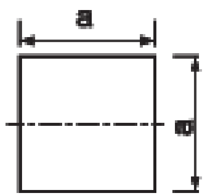
$$M = \frac{EI}{R}$$

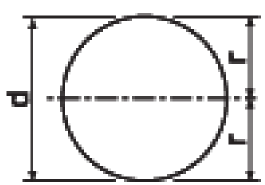
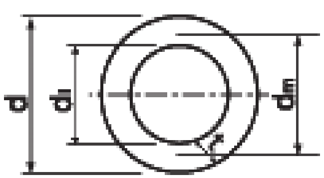
$Z = \dfrac{I}{y}$ = Section modulus

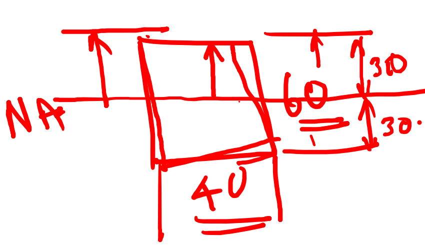
EI — flexural rigidity

BEAM BENDING

L = overall length W = point load, M = moment w = load per unit length	End Slope	Max Deflection	Max bending moment
	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$	M
	$\frac{WL^2}{2EI}$	$\frac{WL^3}{3EI}$	WL
	$\frac{wL^3}{6EI}$	$\frac{wL^4}{8EI}$	$\frac{wL^2}{2}$
	$\frac{ML}{2EI}$	$\frac{ML^2}{8EI}$	M
	$\frac{WL^2}{16EI}$	$\frac{WL^3}{48EI}$	$\frac{WL}{4}$
	$\frac{wL^3}{24EI}$	$\frac{5wL^4}{384EI}$	$\frac{wL^2}{8}$
 $a \le b,\quad c = \sqrt{\frac{1}{3}b(L+a)}$	$\theta_B = \frac{Wac^2}{2LEI}$ $\theta_A = \frac{L+b}{L+a}\,\theta_B$	$\frac{Wac^3}{3LEI}$ (at position c)	$\frac{Wab}{L}$ (under load)

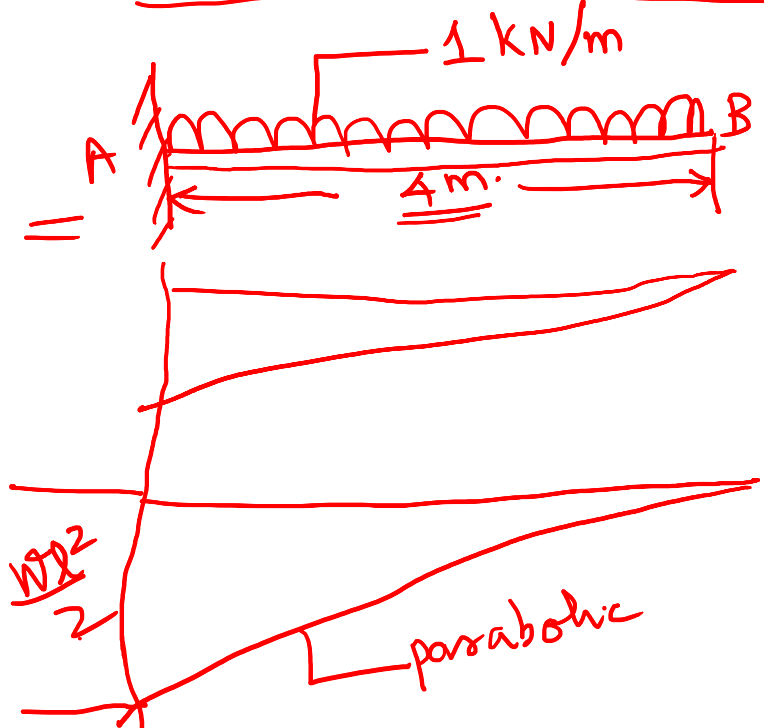
Cross-section shape	Moment of inertia of area	Modulus of section Z
Rectangle 	$I=\frac{bh^3}{12}$	$Z=\frac{bh^2}{6}$
Square 	$I=\frac{a^4}{12}$	$Z=\frac{a^3}{6}$

Cross-section shape	Moment of inertia of area	Modulus of section Z
Circle 	$I=\frac{\pi d^4}{64}=\frac{\pi r^4}{4}$	$Z=\frac{\pi d^3}{32}=\frac{\pi r^3}{4}$
Hollow circle 	$t=\frac{\pi(d^4-d_1^4)}{64}$ Thin wall $I\fallingdotseq\frac{\pi}{8}t\cdot d_m^3$	$Z=\frac{\pi(d^4-d_1^4)}{32d}$ Thin wall $Z\fallingdotseq\frac{\pi}{4}t\cdot d_m^2$



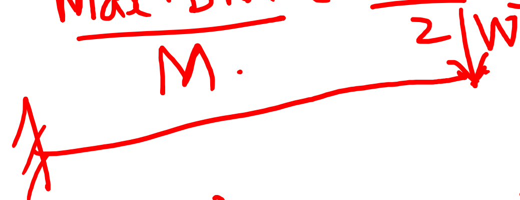
$$\frac{M}{I} = \frac{\sigma}{y} \qquad \sigma = \frac{M}{I}\,y$$

- A cantilever 4 metres long is subjected to a UDL of 1 kN per metre run over the entire span. The section of the cantilever is 40 mm wide and 60 mm deep. Determine the bending stress produced. What point load may be applied at the free end to produce the same bending stress?



$$\text{Max. BM} = \frac{wl^2}{2} = \frac{1\times10^3 \times 4^2}{2}$$

M.



$$= 8\times10^3 \text{ N m.}$$

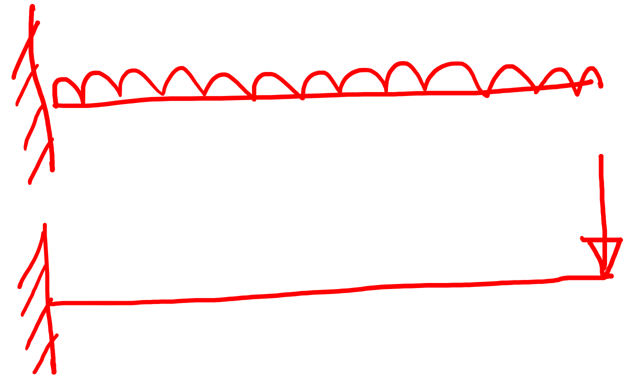
$$= 8\times10^6 \text{ N mm.}$$

$$I = \frac{bd^3}{12} = \frac{40\times60^3}{12} = 72\times10^4 \text{ mm}^4.$$

$$y = \frac{60}{2} = 30 \text{ mm}$$

$$\sigma = \frac{M}{I}\,y = \frac{8\times10^6}{72\times10^4}\times30 = (333.3 \text{ MPa})$$

$$\sigma = 333.3 \text{ MPa}$$



$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{I}{y}\sigma$$

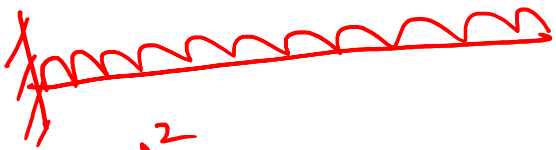
$$Wl = \frac{72 \times 10^4}{30} \times 333.3$$

$$Wl = 8 \times 10^6 \text{ N-mm}$$

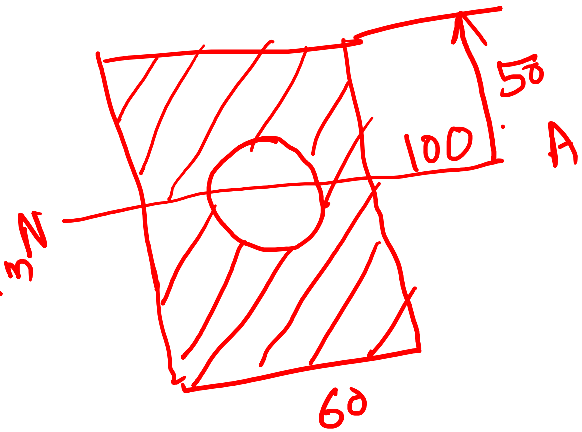
$$W = \frac{8 \times 10^6}{4000} = 2000 \text{ N}$$

$$= 2 \text{ kN}$$

- A cantilever 3 m long carries a UDL of 1 kN per metre run over the whole span. The cross section of the beam is rectangular 60 mm wide and 100 mm deep with a circular hole of 20 mm diameter at the centre. Determine the maximum bending stress induced in the beam



$$M = \frac{wl^2}{2} = \frac{1\times10^3 \times 3^2}{2} = 4.5\times10^6 \text{ N-mm}$$

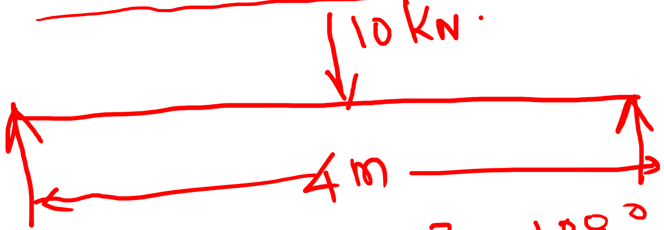


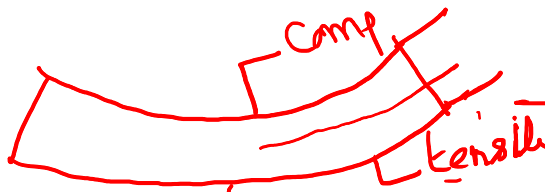
$$I = \frac{60\times100^3}{12} - \frac{\pi}{64}(20)^4 = 499.21\times10^4 \text{ mm}^4.$$

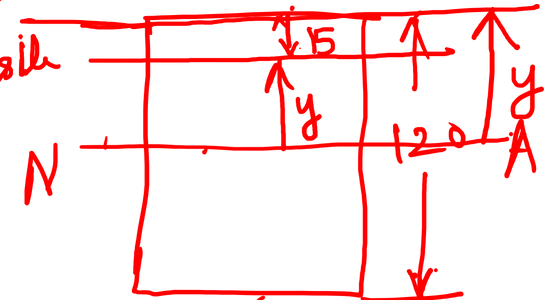
$$y = 50 \text{ mm}.$$

$$\sigma = \frac{M}{I}y = 45.07 \text{ N/mm}^2.$$

- A rectangular beam 120 mm deep and 60 mm wide is simply supported over a span of 4 m and carries a central load of 10 kN. Determine the maximum fibre stress in the beam. Also calculate the stress at a fibre 15 mm from the top surface of the beam.







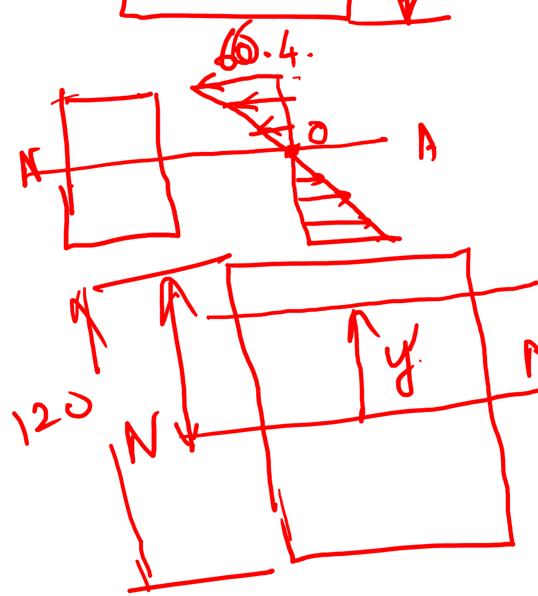
$$M = \frac{Wl}{4} = \frac{10\times10^3 \times 4000}{4} = 10\times10^6 \text{ N-mm}$$

$$I = \frac{bd^3}{12} = \frac{60\times120^3}{12} = 864\times10^4 \text{ mm}^4$$

$y = 60$ mm

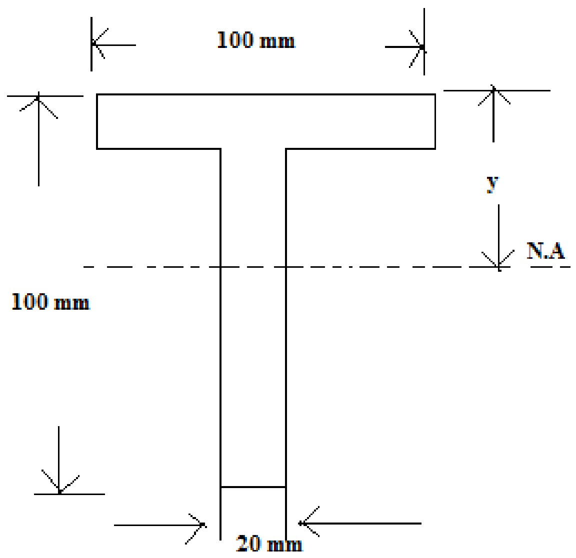
$$\sigma = \frac{M}{I}y = 69.4 \text{ N/mm}^2$$

$y' = 45$ mm



$$\sigma' = \frac{M}{I}y' = 52.08 \text{ N/mm}^2$$

- A cast iron beam is of T-section as shown in Fig. the beam is simply supported on a span of 8 m. the beam carries a UDL of 1.5 kN/m length on the entire span. Determine the max. tensile and max. compressive stresses.



- A 100 mm x 200 mm rolled steel joist of I-section has flanges 12 mm thick and web 10 mm thick. Find the safe uniformly distributed load that this section can carry over a span of 6 m if the permissible skin stress is limited to 160 N/mm^2.

Flitched beam or composite beam

Let E_1 = Young's modulus of steel plate

I_1 = Moment of inertia of steel about N.A.

M_1 = Moment of resistance of steel

E_2 = Young's modulus of wood

I_2 = M.O.I. of wood about N.A.

M_2 = Moment of resistance of wood.

Strain in steel at a distance y from N.A.

$$= \frac{\text{Stress}}{E} = \frac{\sigma_1}{E_1}$$

($\because$ Stress in steel = σ_1)

Strain in wood at a distance y from N.A.

$$= \frac{\sigma_2}{E_2}$$

But strain at the common surface is same

$$\therefore \quad \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \qquad \text{...(7.11)}$$

$$\sigma_1 = \frac{E_1}{E_2} \times \sigma_2$$

$$= m \times \sigma_2 \qquad \text{...}(i)$$

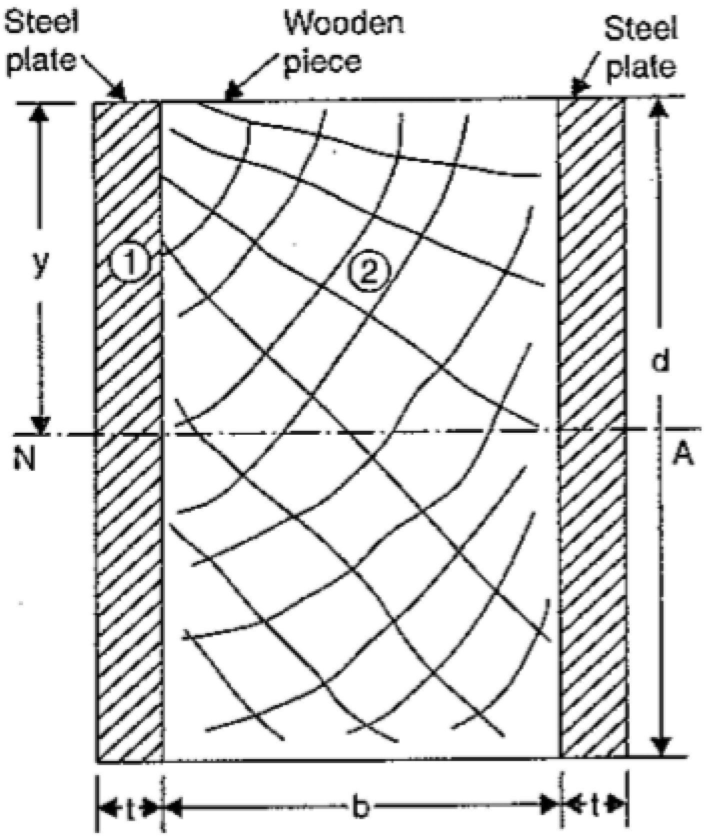


Fig. 7.27 (*a*)

Flitched beam or composite beam

where $m = \frac{E_1}{E_2}$ and is known as modular ratio between steel and wood.

Using the relation $\frac{M}{I} = \frac{\sigma}{y}$, we get

$$M = \frac{\sigma}{y} \times I$$

Hence moment of resistance of steel and wood are given by,

$$M_1 = \frac{\sigma_1}{y} \times I_1 \quad \text{and} \quad M_2 = \frac{\sigma_2}{y} \times I_2$$

∴ Total moment of resistance of the composite section,

$$M = M_1 + M_2$$

$$= \frac{\sigma_1}{y} \times I_1 + \frac{\sigma_2}{y} \times I_2$$

$$= \frac{m\sigma_2 \times I_1}{y} + \frac{\sigma_2}{y} \times I_2 \qquad (\because \ \sigma_1 = m\sigma_2 \text{ from equation } i)$$

$$= \frac{\sigma_2}{y} [mI_1 + I_2] \qquad \text{...(7.12)}$$

Flitched beam or composite beam

In equation (7.12) $I_2 + mI_1$ can be treated as equivalent moment of inertia of the cross-section, as if all made of material 2 (*i.e.*, wood) which will give the same amount of resistance as the composite beam. Let this be denoted by I.

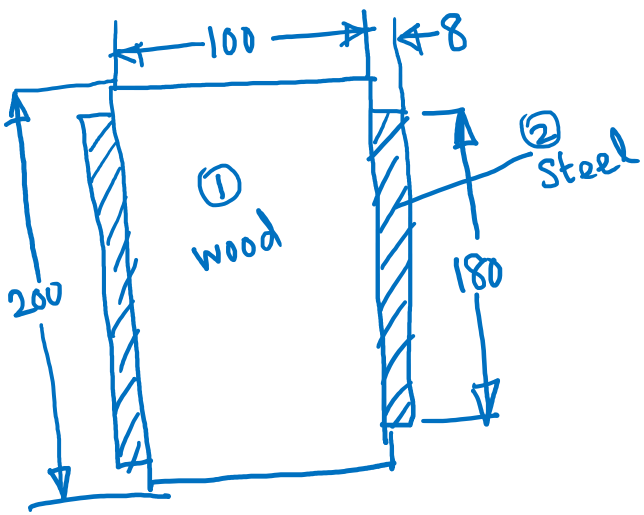
$$\therefore \quad I = mI_1 + I_2 \qquad \text{...(7.13)}$$

Then

$$M = \frac{\sigma_2}{y} \times I \qquad \text{...(7.14)}$$

Flitched beam or composite beam

A composite beam consists of a wooden joist *10 cm* wide, *20 cm* deep strengthened by two steel plates *8 mm* thick and *18 cm* deep placed symmetrically one on either side of the joist. If the stresses in wood and steel are not to exceed in wood and steel are not to exceed *7.5 MPa* and *140 MPa*, find the moment of resistance of the section of the beam. Take the modulus ratio as 20.



Equivalent Moment of Inertia:

$$I = I_1 + m I_2$$

$$= \frac{100 \times 200^3}{12} + 20\left[2 \times \frac{8 \times 180^3}{12}\right] = 2.22 \times 10^8 \text{ mm}^4$$

Moment of Resistance:

$$M = \frac{\sigma_{1max}}{y_{1max}} \times I = \frac{7.5}{100} \times 2.22 \times 10^8$$

$$= 16.65 \times 10^6 \text{ N-mm}$$

Flitched beam or composite beam

$$\sigma_{2max} = m \frac{y_{2max}}{y_{1max}} \sigma_{1max}$$

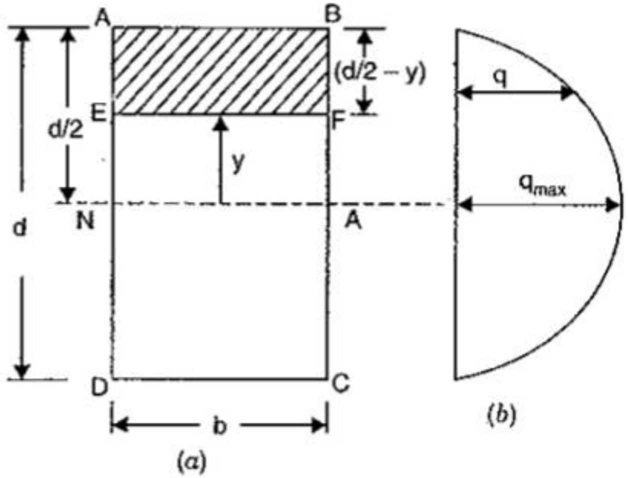
$$= 20 \times \frac{90}{100} \times 7.5$$

$$= 135 \ N/mm^2$$

Which is less than the permissible value ($140 \ N/mm^2$)

Shear stresses

- $\tau = \frac{FA\bar{y}}{Ib} = \frac{QA\bar{y}}{Ib} = \frac{QV}{Ib}$



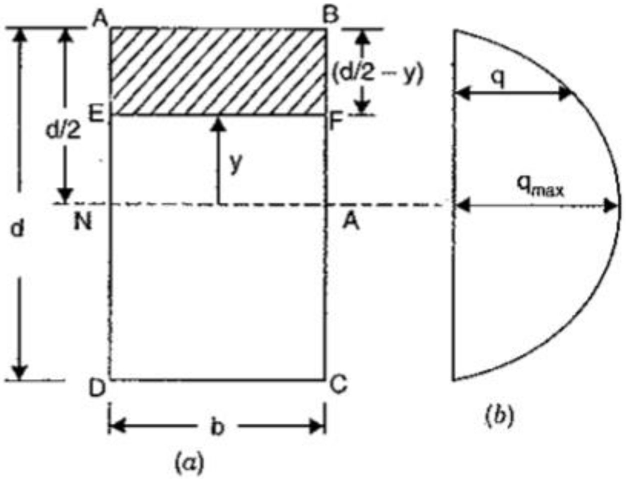
- Where
- F or Q is shear force
- $A = Area\ of\ shaded\ portion$
- $\bar{y} = distance\ from\ NA\ to\ the\ centroid\ of\ shaded\ area$
- $V = A\bar{y} =$ moment of shaded area about NA
- $I = Moment\ of\ Inertia\ of\ the\ entire\ section\ about\ NA$
- $b = width\ of\ the\ fibre$

Shear stresses

$$\tau = F \cdot \frac{A\bar{y}}{b \times I}$$

where A = Area of the section above y (*i.e.*, shaded area $ABFE$)

$$= \left(\frac{d}{2} - y\right) \times b$$



$\bar{y}$ = Distance of the C.G. of area A from neutral axis

$$= y + \frac{1}{2}\left(\frac{d}{2} - y\right) = y + \frac{d}{4} - \frac{y}{2} = \frac{y}{2} + \frac{d}{4} = \frac{1}{2}\left(y + \frac{d}{2}\right)$$

b = Actual width of the section at the level EF

I = M.O.I. of the whole section about N.A.

Substituting these values in the above equation, we get

$$\tau = \frac{F.\left(\frac{d}{2} - y\right) \times b \times \frac{1}{2}\left(y + \frac{d}{2}\right)}{b \times I}$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

Shear stresses

Problem 8.1. *A wooden beam 100 mm wide and 150 mm deep is simply supported over a span of 4 metres. If shear force at a section of the beam is 4500 N, find the shear stress at a distance of 25 mm above the N.A.*

Sol. Given :

Width, $b = 100$ mm

Depth, $d = 150$ mm

Shear force, $F = 4500$ N

Let τ = Shear stress at a distance of 25 mm above the neutral axis.

Using equation (8.1), we get

$$\tau = F . \frac{A\bar{y}}{I . b} \quad ...(i)$$

where A = Area of the beam above y_1

$= 100 \times 50 = 5000 \text{ mm}^2$

(Shaded area of Fig. 8.2)

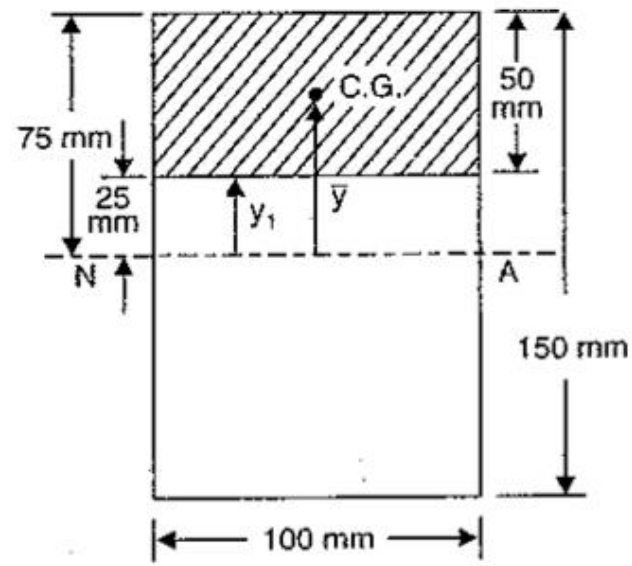


Fig. 8.3

Shear stresses

$\bar{y}$ = Distance of the C.G. of the area A from neutral axis

$$= 25 + \frac{50}{2} = 50 \text{ mm}$$

I = M.O.I. of the total section

$$= \frac{bd^3}{12}$$

$$= \frac{100 \times 150^3}{12} = 28125000 \text{ mm}^4$$

b = Actual width of section at a distance y_1 from N.A. = 100 m

Substituting these values in the above equation (i), we get

$$\tau = \frac{4500 \times 5000 \times 50}{28125000 \times 100} = \mathbf{0.4 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Shear stresses

Problem 8.9. *The shear force acting on a section of a beam is 50 kN. The section of the beam is of T-shaped of dimensions 100 mm × 100 mm × 20 mm as shown in Fig. 8.12. The moment of inertia about the horizontal neutral axis is 314.221 × 10^4 mm^4. Calculate the shear stress at the neutral axis and at the junction of the web and the flange.*

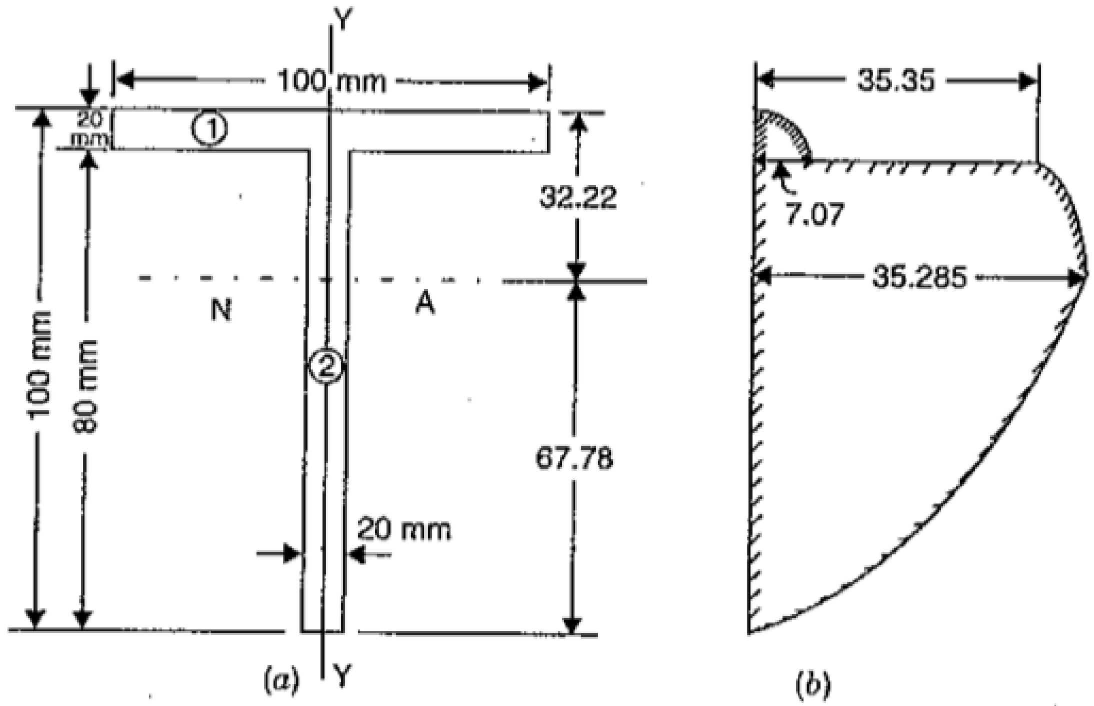


Fig. 8.12

Shear stresses

Fig. 8.12

Sol. Given :

Shear force, $F = 50 \text{ kN} = 50000 \text{ N}$

Moment of inertia about N.A.,

$$I = 314.221 \times 10^4 \text{ mm}^4.$$

First calculate the position of neutral axis. This can be obtained if we know the position of C.G. of given T-section. The given section is symmetrical about the axis *Y-Y* and hence the C.G. of the section will lie on *Y-Y* axis.

Let y^* = Distance of the C.G. of the section from the top of the flange.

Then
$$y^* = \frac{A_1 y_1 + A_2 y_2}{(A_1 + A_2)}$$

$$= \frac{(100 \times 20) \times 10 + (20 \times 80) \times \left(20 + \frac{80}{2}\right)}{(100 \times 10) + (10 \times 90)}$$

$$= \frac{20000 + 96000}{2000 + 1600} = 32.22.$$

Hence, neutral axis will be at a distance of 32.22 mm from the top of the flange as shown in Fig. 8.12 (*a*).

Shear stresses

Shear stress distribution in the flange

Now the shear stress at the top edge of the flange, and bottom of the web is zero.

Shear stress in the flange just at the junction of the flange and web is given by,

$$\tau = \frac{F \times A\bar{y}}{I \times b}$$

where $A = 100 \times 20 = 2000 \text{ mm}^2$

$\bar{y}$ = Distance of C.G. of the area of flange from N.A.

$$= 32.22 - \frac{20}{2} = 22.22 \text{ mm}$$

b = Width of flange = 100 mm

$$\therefore \quad \tau = \frac{50000 \times 2000 \times 22.22}{314.221 \times 10^4 \times 100} = 7.07 \text{ N/mm}^2.$$

Shear stresses

Shear stress distribution in the web

The shear stress in the web just at the junction of the web and flange will suddenly increase from 7.07 N/mm^2 to $7.07 \times \frac{100}{20} = 35.35$ N/mm^2. The shear stress will be maximum at N.A. Hence shear stress at the N.A. is given by

$$\tau = \frac{F \times A\bar{y}}{I \times b}$$

where $A\bar{y}$ = Moment of the above N.A. about N.A.

= Moment of area of flange about N.A. + Moment of area of web about N.A.

$$= 20 \times 100 \times (32.22 - 10) + 20 \times (32.22 - 10) \times \frac{22.22}{2}$$

$$= 44440 + 4937.28 = 49377.284 \text{ mm}^2$$

$b = 20$ mm

$$\therefore \quad \tau = \frac{50000 \times 49377.284}{314.221 \times 10^4 \times 20} = 39.285 \text{ N/mm}^2$$

Now the shear stress distribution diagram can be drawn as shown in Fig. 8.12 (*b*).

Shear stresses

Problem 8.10. *The shear force acting on a beam at an I-section with unequal flanges is 50 kN. The section is shown in Fig. 8.13. The moment of inertia of the section about N.A. is 2.849×10^4. Calculate the shear stress at the N.A. and also draw the shear stress distribution over the depth of the section.*

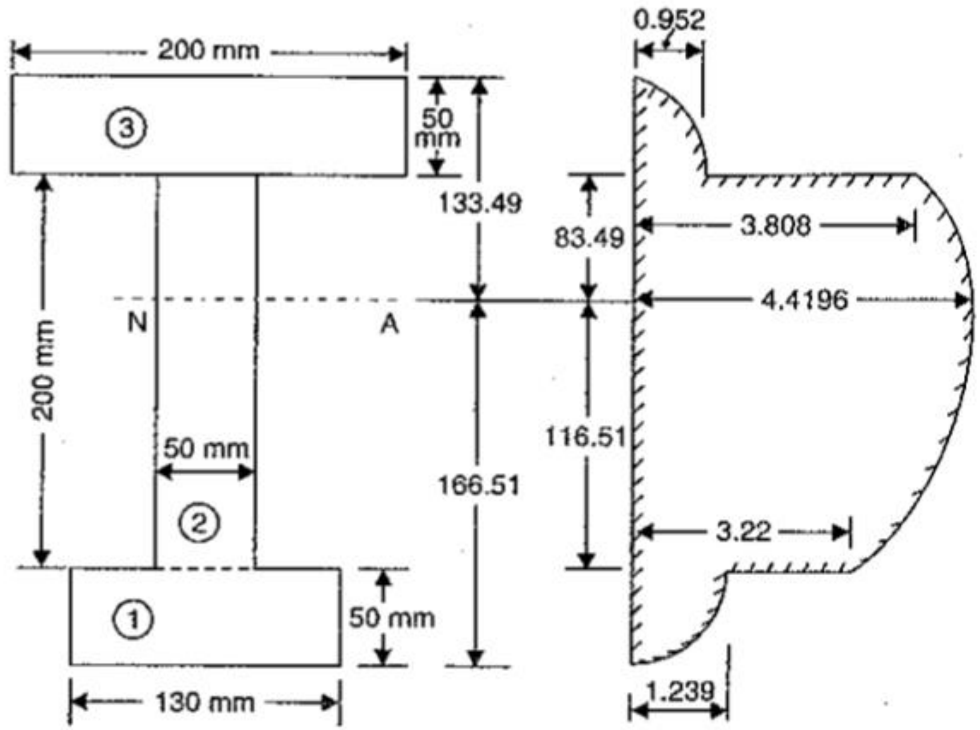


Fig. 8.13

Sol. Given :

Shear force, $F = 50 \text{ kN} = 50{,}000 \text{ N}$

Moment of inertia about N.A.,

$$I = 2.849 \times 10^8 \text{ mm}^4.$$

Let us first calculate the position of N.A. This is obtained if we know the position of the C.G. of the given I-section. Let y^* is the distance of the C.G. from the bottom face. Then

$$y^* = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{(A_1 + A_2 + A_3)}$$

where A_1 = Area of bottom flange

$= 130 \times 50 = 6500 \text{ mm}^2$

A_2 = Area of web $= 200 \times 50 = 10000 \text{ mm}^2$

A_3 = Area of top flange $= 200 \times 50 = 10000 \text{ mm}^2$

y_1 = Distance of C.G. of A_1 from bottom face

$$= \frac{50}{2} = 25 \text{ mm}$$

y_2 = Distance of C.G. of A_2 from bottom face

$$= 50 + \frac{200}{2} = 150 \text{ mm}$$

y_3 = Distance of C.G. of A_3 from bottom face

$$= 50 + 200 + \frac{50}{2} = 275 \text{ mm}$$

$$\therefore \quad y^* = \frac{6500 \times 25 \times 10000 \times 150 + 10000 \times 275}{6500 + 10000 + 10000} = 166.51 \text{ mm}$$

Hence N.A. is at a distance of 166.51 mm from the bottom face (or $300 - 166.51 = 133.49$ mm from upper top fibre).

Shear stress distribution

(*i*) Shear stress at the extreme edges of the flanges is zero.

(*ii*) The shear stress in the upper flange just at the junction of upper flange and web is given by,

$$\tau = \frac{F \times A\bar{y}}{I \times b}$$

where $A\bar{y}$ = Moment of the area of the upper flange about N.A.

= Area of upper flange × Distance of the C.G. of upper flange from N.A.

$= (200 \times 50) \times (133.49 - 25) = 1084900$

b = Width of upper flange = 200 mm

$$\therefore \quad \tau = \frac{50000 \times 1084900}{2.849 \times 108 \times 200} = 0.9520 \text{ N/mm}^2.$$

(*iii*) The shear stress in the web just at the junction of the web and upper flange will suddenly increase from 0.952 to $0.952 \times \frac{200}{50} = 3.808 \text{ N/mm}^2$.

(*iv*) The shear stress will be maximum at the N.A. This is given by

$$\tau_{max} = \frac{F \times A\bar{y}}{I \times b}$$

where $A\bar{y}$ = Moment of total area (about N.A.) about N.A.

= Moment of area of upper flange about N.A. + Moment of area of web about N.A.

$$= 200 \times 50 \times (133.49 - 25) + (133.49 - 50) \times 50 \times \frac{(133.49 - 50)}{2}$$

$$= 1084900 + 174264.5 = 1259164.5$$

and $b = 50$ mm

$$\therefore \quad \tau_{max} = \frac{50000 \times 1259164.5}{2.849 \times 10^8 \times 50} = 4.4196 \text{ N/mm}^2.$$

(*v*) The shear stress in the lower flange just at the junction of the lower flange and the web is given by

$$\tau = \frac{F \times A\bar{y}}{I \times b}$$

where $A\bar{y}$ = Moment of the area of the lower flange about N.A.

$$= 130 \times 50 \times (166.51 - 25) = 918125$$

b = Width of lower flange = 130 mm

$$\therefore \quad \tau = \frac{50000 \times 918125}{2.849 \times 10^8 \times 130} = 1.239 \text{ N/mm}^2.$$

(*vi*) The shear stress in the web just at the junction of the web and lower flange will suddenly increase from 1.239 to $\frac{1.239 \times 130}{50} = 3.22 \text{ N/mm}^2$.

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Thank you

STRENGTH OF MATERIALS FOR MECHANICAL ENGINEERS (CE8395)

Unit-3

(TORSION)

by

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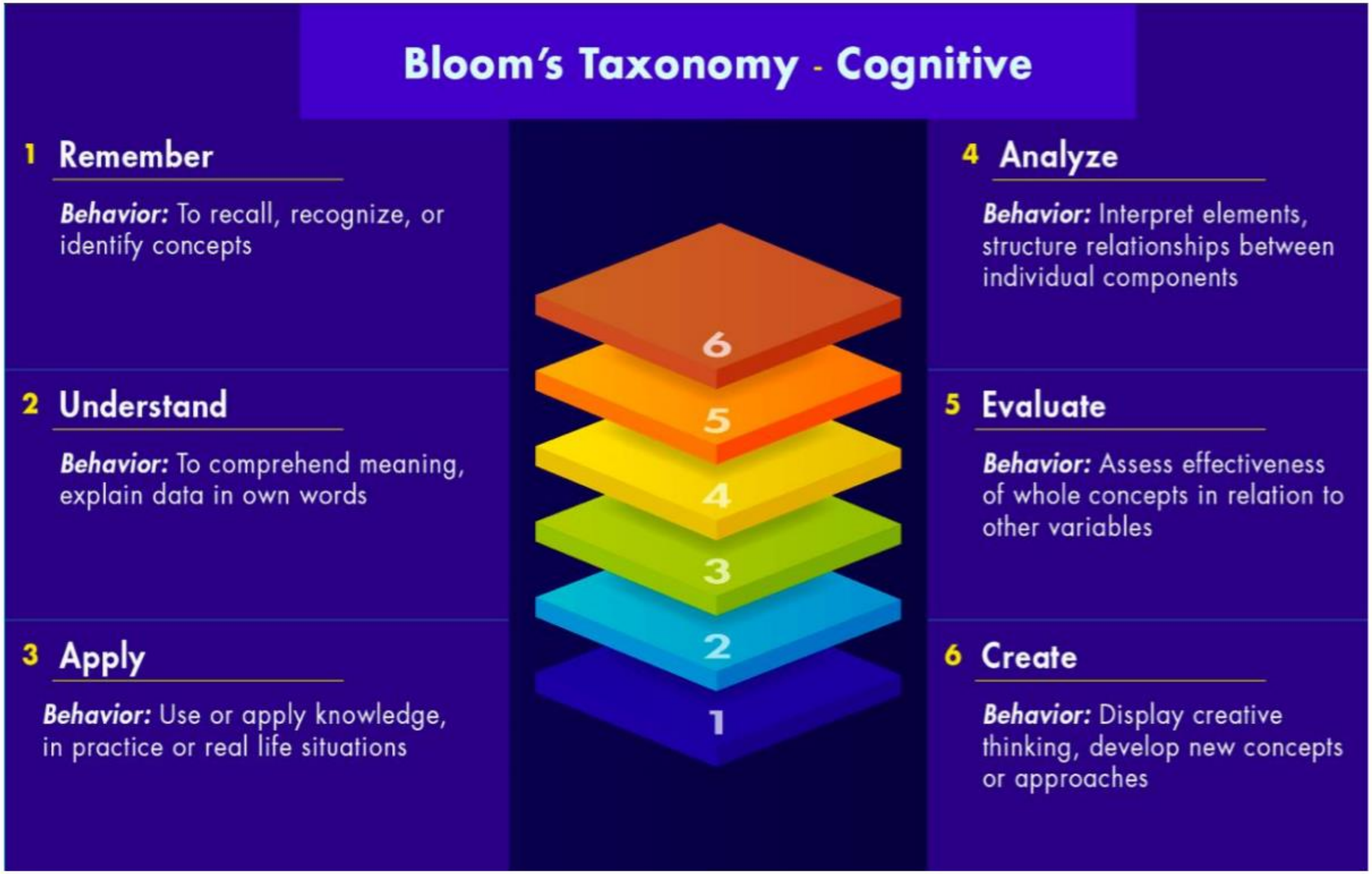
Course Objectives:

1. To understand the concepts of stress, strain, principal stresses and principal planes.
2. To study the concept of shearing force and bending moment due to external loads in determinate beams and their effect on stresses.
3. **To determine stresses and deformation in circular shafts and helical spring due to torsion.**
4. To compute slopes and deflections in determinate beams by various methods.
5. To study the stresses and deformations induced in thin and thick shells.

Course outcomes

Students will be able to

1. Understand the concepts of stress and strain in simple and compound bars, the importance of principal stresses and principal planes.
2. Understand the load transferring mechanism in beams and stress distribution due to shearing force and bending moment.
3. **Apply basic equation of simple torsion in designing of shafts and helical spring**
4. Calculate the slope and deflection in beams using different methods.
5. Analyze and design thin and thick shells for the applied internal and external pressures.



Bloom's Taxonomy - Cognitive
1 Remember
Behavior: To recall, recognize, or identify concepts
2 Understand
Behavior: To comprehend meaning, explain data in own words
3 Apply
Behavior: Use or apply knowledge, in practice or real life situations
4 Analyze
Behavior: Interpret elements, structure relationships between individual components
5 Evaluate
Behavior: Assess effectiveness of whole concepts in relation to other variables
6 Create
Behavior: Display creative thinking, develop new concepts or approaches
6
5
4
3
2
1

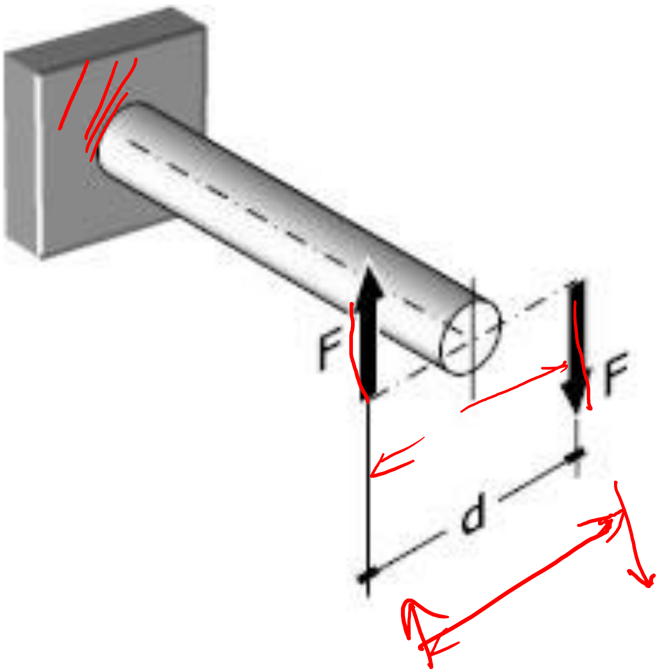
Syllabus Unit-3

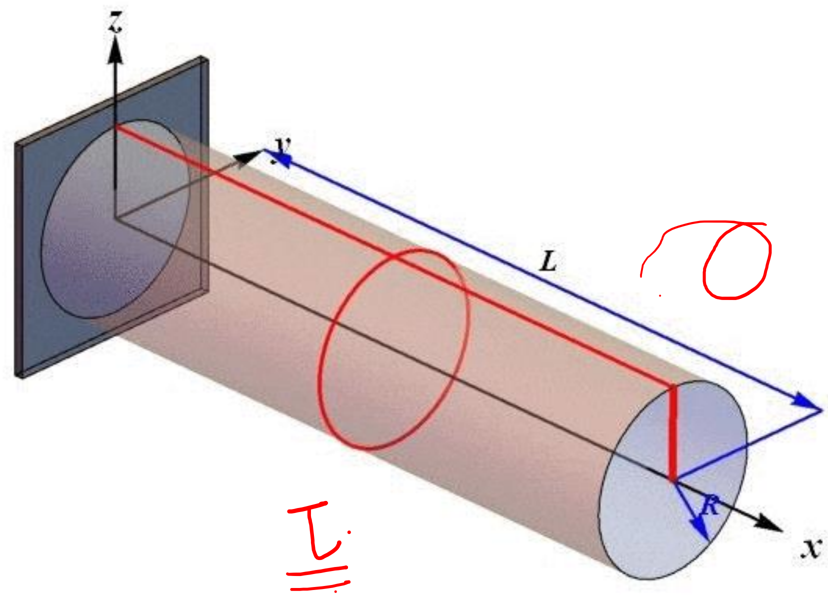
UNIT III Torsion

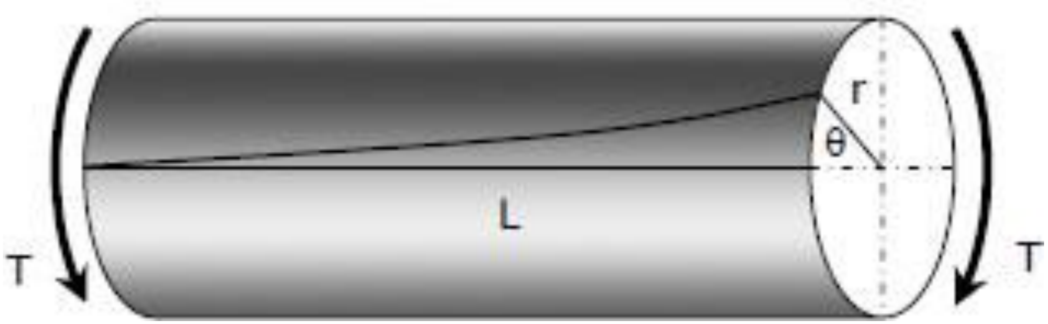
Torsion formulation stresses and deformation in circular and hollows shafts – Stepped shafts– Deflection in shafts fixed at the both ends

Stresses in helical springs – Deflection of helical springs, carriage springs

Torsion







Torsion equation

General Torsion Equation (Shafts of circular cross-section)

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\,\theta}{L}$$

1. For Solid Shaft

$$J = \frac{\pi}{2} r^4 = \frac{\pi d^4}{32}$$

2. For Hollow Shaft

$$J = \frac{\pi}{2}(r_1^4 - r_2^4)$$

$$= \frac{\pi}{32}(d_1^4 - d_2^4)$$

T	=	torque or twisting moment in newton metres
J	=	polar second moment of area of cross-section about shaft axis.
τ	=	shear stress at outer fibres in pascals
r	=	radius of shaft in metres
G	=	modulus of rigidity in pascals
θ	=	angle of twist in radians
L	=	length of shaft in metres
d	=	diameter of shaft in metres

Strength of shafts

Maximum torque or power the shaft can transmit from one pulley to another, is called strength of shaft.

(a) For solid circular shafts:

Maximum torque (T)is given by :

$$T = \frac{\pi}{16} \times \tau \times D^3$$

where, D = dia. of the shaft

τ=shear stress in the shaft

b) For Hollow shafts

$$= \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0} \right]$$

Power transmitted

16.5. POWER TRANSMITTED BY SHAFTS

Once the expression for torque (T) for a solid or a hollow shaft is obtained, power transmitted by the shafts can be determined.

Let N = r.p.m. of the shaft

T = Mean torque transmitted in N-m

ω = Angular speed of shaft.

Then
$$\text{Power} = \frac{2\pi N T^*}{60} \text{ watts} \quad \text{...(16.7)}$$

$$= \omega \times T \quad \left(\because \frac{2\pi N}{60} = \omega\right)$$

$$= T \times \omega \quad \text{...[16.7 (A)]}$$

If the torque fluctuates, the greatest torque must be used for evaluating the maximum shear stress due to torsion
On the other hand, for calculating power, mean torque should be used

Torsional rigidity

Let a twisting moment T produces a twist of θ radians in a shaft of length L. Using equation (16.9), we have

$$\frac{T}{J} = \frac{C\theta}{L} \quad \text{or} \quad C \times J = \frac{T \times L}{\theta}$$

But $\quad C \times J = \text{Torsional rigidity}$

$\therefore \quad \text{Torsional rigidity} = \dfrac{T \times L}{\theta}$

If L = one metre and θ = one radian

Then torsional rigidity = Torque.

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{T}{\theta} = \frac{CJ}{l}$$

Torsional stiffness $= \dfrac{T}{\theta}$

Torsional rigidity $= CJ$

- **Torsional rigidity:** It is defined as the product of modulus of rigidity and polar moment of inertia.

K = G x J

- S.I. unit of torsional rigidity is Nm^2
- **Nm/radian** is the S.I. unit of torsional stiffness. **Torsional stiffness** is defined as the torque required to produce unit angle of twist.

Polar modulus

- **Polar modulus** is defined as the ratio of the **polar** moment of inertia to the radius of the shaft.
- It is also called as **torsional** section **modulus**.
- It is denoted by Z_p.
- Polar Modulus (Z_p) is a Direct Measure of Torsional strength of a Shaft.

$$\frac{T}{J} = \frac{\tau}{r} \quad \text{or} \quad T = \tau \frac{J}{r} = \tau Z_p \qquad \left(Z_p = \frac{J}{r} \right)$$

Problem - 1

Problem 16.1. *A solid shaft of 150 mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the maximum shear stress induced to the shaft is 45 N/mm².*

Sol. Given :

Diameter of the shaft, $D = 150$ mm

Maximum shear stress, $\tau = 45\ N/mm^2$

Let T = Maximum torque transmitted by the shaft.

Using equation (16.4), $T = \frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \times 45 \times 150^3$

$= 29820586$ N-mm = **29820.586 N-m. Ans.**

Problem - 2

Problem 16.2. *The shearing stress is a solid shaft is not to exceed 40 N/mm² when the torque transmitted is 20000 N-m. Determine the minimum diameter of the shaft.*

Sol. Given :

Maximum shear stress, $\tau = 40 \text{ N/mm}^2$

Torque transmitted, $T = 20000 \text{ N-m} = 20000 \times 10^3 \text{ N-mm}$

Let D = Minimum diameter of the shaft in mm.

Using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$D = \left(\frac{16T}{\pi\tau}\right)^{1/3} = \left(\frac{16 \times 20000 \times 10^3}{\pi \times 40}\right)^{1/3} = 136.2 \text{ mm.} \quad \textbf{Ans.}$$

Problem - 3

Problem 16.3. *In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively, the shear stress is not to exceed 40 N/mm². Find the maximum torque which the shaft can safely transmit.*

Sol. Given :

Outer diameter, $D_0 = 20 \text{ cm} = 200 \text{ mm}$

Inner diameter, $D_i = 10 \text{ cm} = 100 \text{ mm}$

Maximum shear stress, $\tau = 40 \text{ N/mm}^2$

Let T = Maximum torque transmitted by the shaft.

Using equation (16.6),

$$T = \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} \times 40 \left[\frac{200^4 - 100^4}{200}\right]$$

$$= \frac{\pi}{16} \times 40 \left[\frac{16 \times 10^8 - 1 \times 10^8}{200}\right] = 58904860 \text{ Nmm}$$

$= 58904.86$ **Nm. Ans.**

Problem - 4

Problem 2. A solid circular shaft is to be designed to transmit 22.5 kW power at 200 r.p.m. If the maximum shear stress is not to exceed 80 N/mm^2 and the angle of twist is not to exceed 1° per metre length, determine the diameter of the shaft. Take modulus of rigidity 80 kN/mm^2.

Solution: $P = 22.5 \text{ kW} = 22.5 \times 10^6$ N-mm/sec.

$$N = 200 \text{ rpm} \qquad \theta = 1° = \frac{\pi}{180} \text{ radians.}$$

$$L = 1000 \text{ mm} \qquad q_s = 80 \text{ N/mm}^2 \qquad G = 80 \times 10^3 \text{ N/mm}^2$$

$$P = \frac{2\pi NT}{60}$$

$$22.5 \times 10^6 = \frac{2 \times \pi \times 200 \times T}{60}$$

or $T = 1074295.86$ N-mm

Let 'd' be the diameter of the shaft.

$$\therefore \qquad J = \frac{\pi}{32} d^4$$

Problem – 4 contd.

From the considerations of shear stress

$$\frac{T}{J} = \frac{q_s}{R}$$

$$\frac{1074295.86}{\frac{\pi}{32} d^4} = \frac{80}{\frac{d}{2}}$$

$$\therefore \quad d = 40.89 \text{ mm}$$

From the consideration of angle of twist

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\frac{1074295.86}{\frac{\pi d^4}{32}} = \frac{80 \times 10^3 \times \frac{\pi}{180}}{1000}$$

$$d^4 = 1074295.86 \times \frac{32}{\pi} \times \frac{1}{80} \times \frac{180}{\pi}$$

$$\therefore \quad d = 52.91 \text{ mm}$$

$\therefore$ **Minimum diameter of the shaft to be used is 52.91 mm.**

Problem - 5

Problem 3. A hollow circular shaft 12 m long is required to transmit 100 kW power when running at a speed of 300 rpm. If the maximum shearing stress allowed in the shaft is 80 N/mm^2 and the ratio of inner diameter to the outer diameter is 0.75, find the dimensions of the shaft and also the angle of twist of one end of the shaft relative to the other end. Modulus of rigidity of the material is 85 kN/mm^2.

Solution:

$$L = 12 \text{ m} = 12000 \text{ mm}$$

$$P = 100 \text{ kW} = 100 \times 10^6 \text{ N-mm/sec}$$

$$N = 300 \text{ rpm}$$

$$q_s = 80 \text{ N/mm}^2$$

and $$G = 85 \times 10^3 \text{ N/mm}^2$$

Let d_1 be the outer diameter and d_2 be the inner diameter

$\therefore$ $$d_2 = 0.75\, d_1, \quad \text{given}$$

Now, $$P = \frac{2\pi NT}{60}$$

i.e. $$100 \times 10^6 = \frac{2\pi \times 300 \times T}{60}$$

$$T = 3183098.8 \text{ N-mm.}$$

Problem – 5 contd.

From the torsion formula,

$$T = J\frac{q_s}{R} = \frac{\pi}{32}\left\{d_1^4 - (0.75\,d_1)^4\right\}\frac{q_s}{\frac{d_1}{2}}$$

we get, $$3183098.8 = \frac{\pi}{16}\{1-(0.75)^4\}\,d_1^3 \times 80$$

$\therefore$ $$d_1^{\,3} = 296436.83$$

$\therefore$ $$d_1 = \mathbf{66.67\ mm.}$$

$\therefore$ $$d_2 = 0.75 \times 66.67 = \mathbf{50\ mm}$$

Again, from torsion formula,

$$\frac{T}{J} = \frac{G\theta}{L}$$

we get, $$\frac{3183098.86}{\frac{\pi}{32}(66.67^4 - 50^4)} = \frac{85\times 10^3\,\theta}{12000}$$

$\therefore$ $$\theta = 0.3389 \text{ radians}$$

Problem – 6

Problem 16.7. *A hollow shaft of external diameter 120 mm transmits 300 kW power at 200 r.p.m. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm^2.* (AMIE, Summer 1990)

Sol. Given :

External dia., $D_0 = 120$ mm

Power, $P = 300$ kW $= 300,000$ W

Speed, $N = 200$ r.p.m.

Max. shear stress, $\tau = 60$ N/mm^2

Let D_i = Internal dia. of shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60} \quad \text{or} \quad 300,000 = \frac{2\pi \times 200 \times T}{60}$$

$$\therefore \quad T = \frac{300,000 \times 60}{2\pi \times 200} = 14323.9 \text{ Nm}$$

$$= 14323.9 \times 1000 \text{ Nmm} = 14323900 \text{ Nmm}$$

Now using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

or
$$14323900 = \frac{\pi}{16} \times 60 \times \frac{(120^4 - D_i^4)}{120}$$

or
$$\frac{14323900 \times 16 \times 120}{\pi \times 60} = 120^4 - D_i^4$$

$$145902000 = 207360000 - D_i^4$$

or
$$D_i^4 = 207360000 - 145902000 = 61458000$$

$$\therefore \quad D_i = (61458000)^{1/4} = 88.5 \text{ mm. } \textbf{Ans.}$$

Replacing of shaft

- When a solid shaft is to be replaced by a hollow shaft or vice-versa, then the power transmitted by the new shaft should always be equal to the power transmitted by the shaft to be replaced

Problem-7

- A solid steel shaft of 50 mm diameter is to be replaced by a hollow steel shaft whose internal diameter is 0.5 times outer diameter. Find the diameters of hollow shaft and percentage saving of weight.

Problem-8

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress τ_w is not to exceed 40 MPa and the angle of twist θ is limited to 6° in a length of 3 m. Use G = 83 GPa.

Solution

Applying Eq. (3.6a) to determine the torque:

$$T = \frac{P}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5 N \cdot m$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{\max} = \frac{Tr}{J} \quad \tau_{\max} = \frac{16T}{\pi d^3} \quad 4 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

Which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

Problem-8 contd..

Apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad\qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{(83\times10^{9})(\pi d^{4}/32)}$$

From which we obtain $d = 48.6\times10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter-namely,

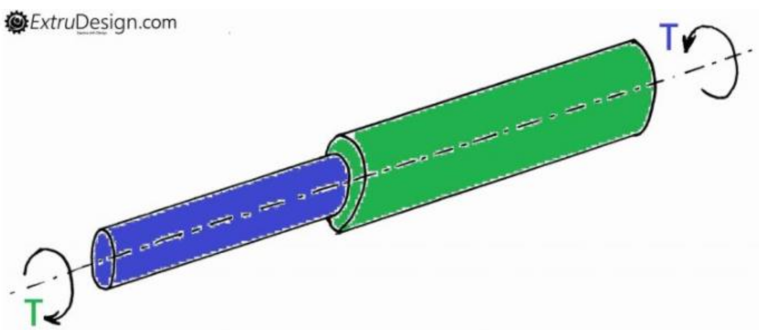
$d = 58.7$ mm. *Answer*

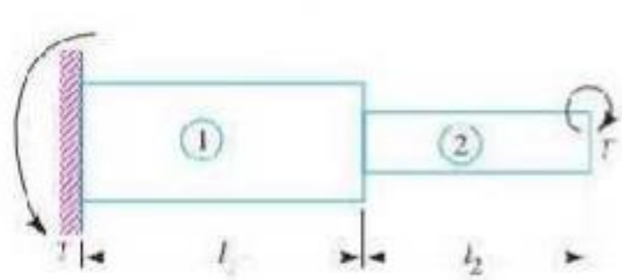
Composite shafts

- When a shaft is having two different diameters/sections it is referred as the composite shaft and it is subjected to torque at the different planes, then the angle of twist in the shaft is to be calculated by considering them in either series or parallel.

Shafts in series

- When a shaft is having two different diameters cross section then two equal torques (**T**) are applied in opposite direction at both ends as shown in the figure. Then the shafts are said to be in series.





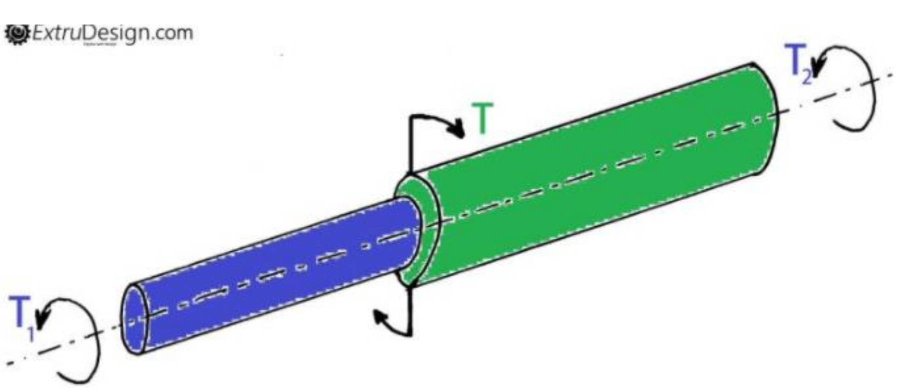
(a) Shafts in series.

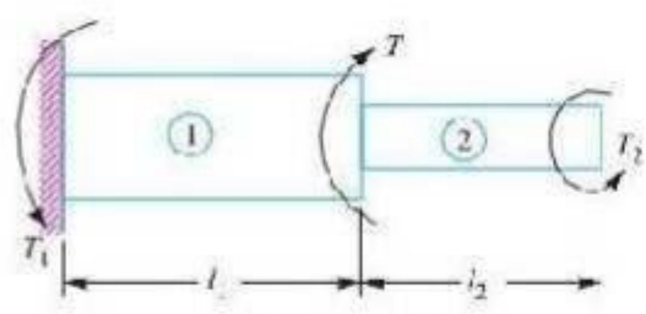
- The angle of twist is the sum of the angle of twist of the two shafts connected in series

$$\theta = \theta_1 + \theta_2 = \frac{T\, l_1}{C_1\ J_1} + \frac{T\, l_2}{C_2\ J_2}$$

Shafts in parallel

- When a shaft is having two different diameters cross section then a torque (**T**) is applied at the centre (Junction of the two different section) and two opposite torques T_1 and T_2 as shown in the figure. Then the shafts are said to be in parallel.



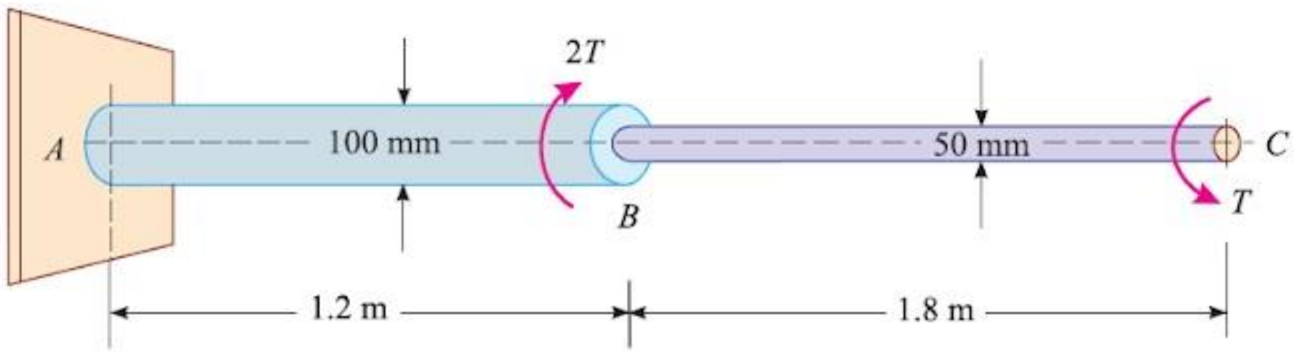


(b) Shafts in parallel.

$$\frac{T_1 l_1}{C_1 J_1} = \frac{T_2 l_2}{C_2 J_2}$$

Problem-9

- The stepped shaft as shown in Fig. is subjected to a torque of T at the free end and a torque of 2T in the opposite direction at the junction of two sizes. What is the total angle of twist at the free end, if the maximum shear stress in the shaft is limited to 70 MN/m^2? Assume the modulus of rigidity to e 84 GN/m^2.



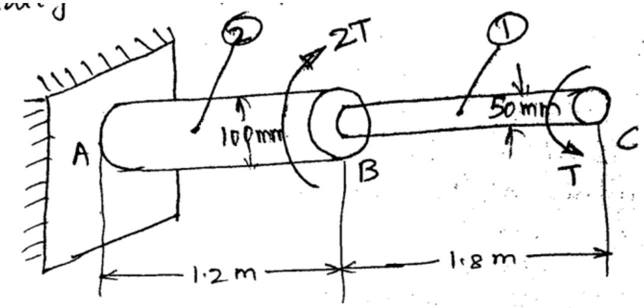


fig.

Soln: Torque on the shaft BC = T KNm.

∴ Torque on shaft AB = T − 2T = −T KNm.
Hence the two shafts are subjected to a torque of same ~~material~~ magnitude but of opposite sense.

For the length of BC:

$$T = \frac{\pi}{16} \tau d^3 = \frac{\pi}{16} \times 70 \times 10^6 \times (0.050)^3 = 1718.1 \text{ Nm}$$

$l = 1.8\,m,$

$$J = \frac{\pi}{32}(0.05)^4 = 6.136 \times 10^{-7} \text{ m}^4$$

$$\theta_1 = \frac{T l_1}{CJ} = \frac{1718.1 \times 1.8}{84 \times 10^9 \times 6.136 \times 10^{-7}} = 0.06 \text{ rad}.$$

For the length AB: Torque = T, $l_2 = 1.2\,m$

$$J_2 = \frac{\pi}{32} \times (0.100)^4 = 9.817 \times 10^{-6} \text{ m}^4$$

$$\theta_2 = \frac{T l_2}{C J_2} = \frac{1718.1 \times 1.2}{84 \times 10^9 \times 9.817 \times 10^{-6}} = 0.0025 \text{ rad}.$$

$$\theta_C = \theta_1 - \theta_2 = 0.06 - 0.0025$$
$$= 0.0575 \text{ rad} = 3.29 \text{ degrees}.$$

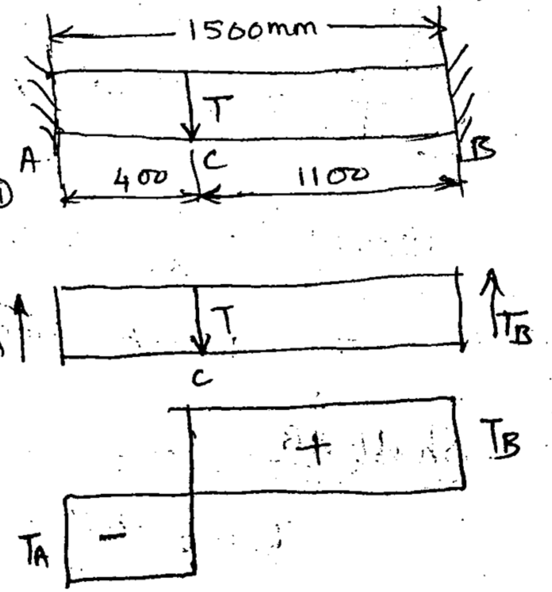
Problem-10

- A twisting moment of 2 kNm is applied to a shaft of 70 mm diameter and 1.5 m length at a distance of 400 mm from one end. The shaft is fixed at both ends. Find the fixing couples at the ends, maximum shear stress setup and the angle of twist of that section where the twisting moment is applied. Take G = 84000 N/mm^2.

10) A twisting moment of 2 kNm is applied to a shaft of 70 mm diameter and 1.5 m length at a distance of 400 mm from one end. The shaft is fixed at both ends. Find the fixing couples at the ends, maximum shear stress set up and the angle of twist of that section where the twisting moment is applied. Take $G = 84000\ N/mm^2$

Soln: The supports are replaced by the fixing couples T_A and T_B.

For equilibrium $T = T_A + T_B$ —— (a)



$$\theta_{AC} = \theta_{BC}$$

$$\frac{T_A \cdot L_{AC}}{GJ} = \frac{T_B \cdot L_{BC}}{CJ}$$

$$T_A \times 400 = T_B \times 1100$$

$$T_A = 2.75\ T_B \quad \text{—— (b)}$$

Now substitute in (a)

$$2 \times 10^6 = 2.75\ T_B + T_B$$

$T_B = 5.33 \times 10^5\ Nmm$, $T_A = 2.75 \times 5.33 \times 10^5 = 14.6 \times 10^5\ Nmm$

$$\theta_{AC} = \frac{14.6 \times 10^5 \times 400}{84000 \times \frac{\pi}{32}(70)^4}$$

$$= \underline{0.0029\ rad}$$

$$\tau_{max} = \frac{16 T_A}{\pi d^3} = \frac{16 \times 14.6 \times 10^5}{\pi \times 70^3} = 21.6\ N/mm^2$$

Springs

- Any elastic member which can deform when loaded and recover its original shape when the load is removed
- Energy absorbing devices
- Examples
  - Ball point pen – compression helical (open coiled helical springs)
  - Spring balance – tensile helical (close coiled helical springs)
  - Writing pad – torsion spring
  - Shock absorber in automobiles, safety valves, clutches

Springs

Close-coiled helical spring

Tension Spring

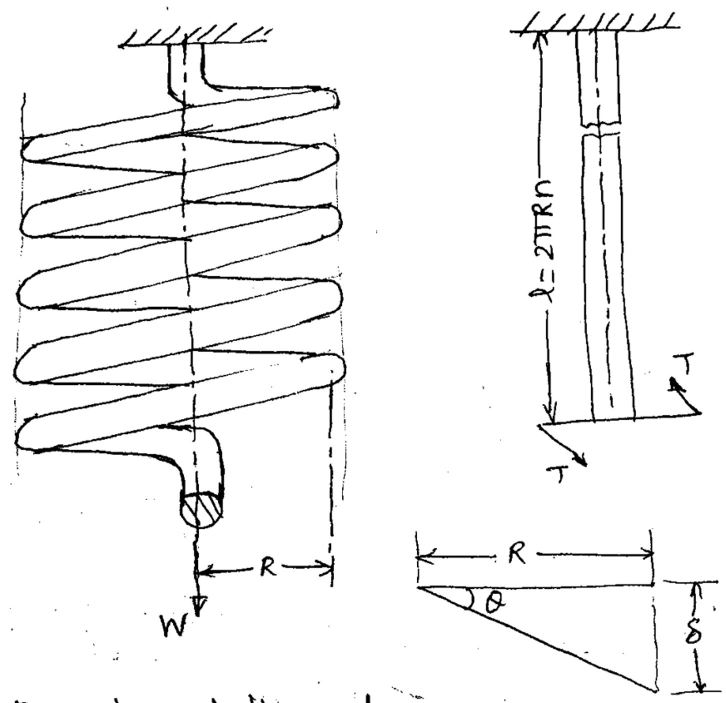
Open-coiled helical spring

Compression Spring

Torsion Spring

Spiral Spring

Close coiled helical spring with axial load



Let R - radius of the coil

d - dia. of the wire of the coil

δ – deflection of coil under the load W

C – modulus of rigidity

n – no. of coils or turns

θ – angle of twist

l – length of wire = $2\pi Rn$

τ – Shear stress and

I_p or J – polar moment of inertia = $\frac{\pi}{32} d^4$

a) Shear Stress τ: from Torsion equation

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{l} \quad ; \quad \frac{T}{J} = \frac{\tau}{R}.$$

$$T = \frac{\tau J}{R} = \frac{\tau \times \frac{\pi}{32} d^4}{d/2} = \tau \times \frac{\pi}{16} d^3$$

$$\tau = \frac{16T}{\pi d^3} \quad \text{or} \quad \boxed{\tau = \frac{16WR}{\pi d^3}} \quad (\because T = WR)$$

b) Deflection δ:

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\theta = \frac{Tl}{CJ} = \frac{WR \times 2\pi R n}{C \times \frac{\pi}{32} d^4} = \frac{64 WR^2 n}{Cd^4}$$

but $\delta = R\theta$

$$\boxed{\delta = \frac{64 WR^3 n}{Cd^4}}$$

c) Energy Stored in a Spring U:

Energy stored $U = \frac{1}{2} \times T \times \theta$

$$= \frac{1}{2} \times WR \times \frac{64 WR^2 n}{Cd^4} = \boxed{\frac{32 W^2 R^3 n}{Cd^4}}$$

$$= \frac{1}{2} \times \frac{1}{2} \frac{16WR}{Cd^3} \times \frac{8WR^2 n}{d} = \frac{1}{4C} \cdot \frac{16WR}{\pi d^3} \times \frac{16WR}{\pi d^3} \left[2\pi R n d^2 \times \frac{\pi}{4} \right]$$

$$\boxed{U = \frac{1}{4C} \times \tau^2 \times \text{volume of wire}}$$

Again energy stored, $U = \frac{1}{2} \times T \times \theta$

$$= \frac{1}{2} \times WR \times \frac{\delta}{R} \quad (\delta = R\theta)$$

$$\boxed{U = \frac{1}{2} W \delta}$$

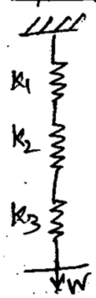
d) Stiffness/Spring rate K

Stiffness is defined as the load required to cause unit deflection.

$$K = \frac{W}{\delta} = \frac{W}{\frac{64WR^3n}{Cd^4}} = \frac{Cd^4}{64R^3n}$$

e) Spring index, C Ratio D/d is called spring index.

f) Springs in series connection

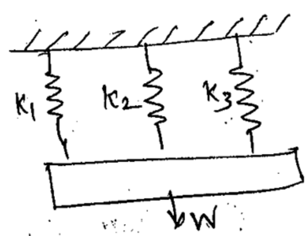


Load is common for all the springs

~~P = P~~ $W = W_1 = W_2 = W_3$

$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

g) Springs in parallel



Deflection is same for all the springs

$$\delta = \delta_1 = \delta_2 = \delta_3$$

$$K = K_1 + K_2 + K_3$$

(1) A close coiled helical spring 8 mm diameter wire with 12 coils of mean diameter 100 mm carries an axial load of 400 N. Find the shear stress induced and deflection caused. What is the strain energy stored. Take $N = 8 \times 10^6 \ N/cm^2$

(APRIL 1995)

Soln:

$d = 8mm, \quad n = 12, \quad D = 100 \ mm \quad W = 400 \ N$

$R = 50 \ mm$

$N \text{ or } C = 8 \times 10^6 \ N/cm^2 = 8 \times 10^{10} \ N/m^2$

Shear stress $\tau = \frac{16WR}{\pi d^3} = \frac{16 \times 400 \times 0.05}{\pi (0.008)^3} = 198.9 \ MN/m^2$

deflection $\delta = \frac{64 W R^3 n}{C d^4} = \frac{64 \times 400 \times (0.05)^3 \times 12}{8 \times 10^{10} \times (0.008)^4} = 0.1171 \ m$

Strain energy $U = \frac{1}{2} \times W \times \delta = \frac{1}{2} \times 400 \times 0.1171 = 23.43 \ Nm.$

or $U = \frac{\tau^2}{4C} \times$ volume of wire

$= \frac{\tau^2}{4C} \times \frac{\pi}{4} d^2 \times 2\pi R n$

$= \frac{(198.9 \times 10^6)^2}{4 \times 8 \times 10^{10}} \times \frac{\pi}{4} (0.008)^2 \times 2\pi (0.05) 12$

$U = 23.43 \ Nm$

(2) A close coiled helical spring made of circular wire is required to absorb 2.5 kN-cm of energy for a deflection of 125 mm and the stress not exceeding 34 MPa. Determine a suitable diameter and length of wire given that mean coil diameter is 150 mm. Take N = 30 GPa. (Oct 95)

Soln: $U = 2.5$ kN-cm $= 2.5 \times \frac{10^3}{100} = 25$ N-m

$\delta = 125$ mm $= 0.125$ m

$\tau = 34$ MPa $= 34 \times 10^6$ N/m^2

$D = 150$ mm, $R = 75$ mm $= 0.075$ m

N or C = 30 GPa $= 30 \times 10^9$ N/m^2

$$U = \frac{1}{2} \times W \times \delta$$

$$25 = \frac{1}{2} \times W \times 0.125$$

$$W = 400 \text{ N}$$

To find dia d

$$\tau = \frac{16WR}{\pi d^3} \; ; \quad 34 \times 10^6 = \frac{16 \times 400 \times 0.075}{\pi d^3}$$

$$d = 0.0165 \text{ m} = 16.5 \text{ mm}$$

To find length ($2\pi R n$)

$$\delta = \frac{64 W R^3 n}{C d^4} \; ; \quad 0.125 = \frac{64 \times 400 \times (0.075)^3 \times n}{30 \times 10^9 \times (0.0165)^4}$$

$$n = 25.7 \text{ coils.}$$

Length of wire required $= 2\pi R n$

$= 2\pi (0.075) \times 25.7$

$= 12.12$ m

(3) A wire of 6mm diameter is used to make a close coiled helical spring with a stiffness of 20 N/mm. The spring has to carry a load of 300 N. The maximum allowable stress in the wire is 90 MPa. Determine the mean radius of the coil and the number of coils required. Shear modulus G = 80 KN/mm². (Oct 97)

Soln: $d = 6\text{mm} = 0.006\text{m}$.

$k = 20\text{ N/mm} = 20\times10^3\text{ N/m}$, $W = 300\text{ N}$

$\tau = 90\text{ MPa} = 90\times10^6\text{ N/m}^2$

$G = 80\text{ KN/mm}^2 = 80\times10^3\times10^6\text{ N/m}^2 = 80\times10^9\text{ N/m}^2$

To find mean radius

$$\tau = \frac{16WR}{\pi d^3};\quad 90\times10^6 = \frac{16\times300\times R}{\pi(0.006)^3}$$

$$R = 0.0127\text{ m} = 12.7\text{ mm}$$

To find no. of coils

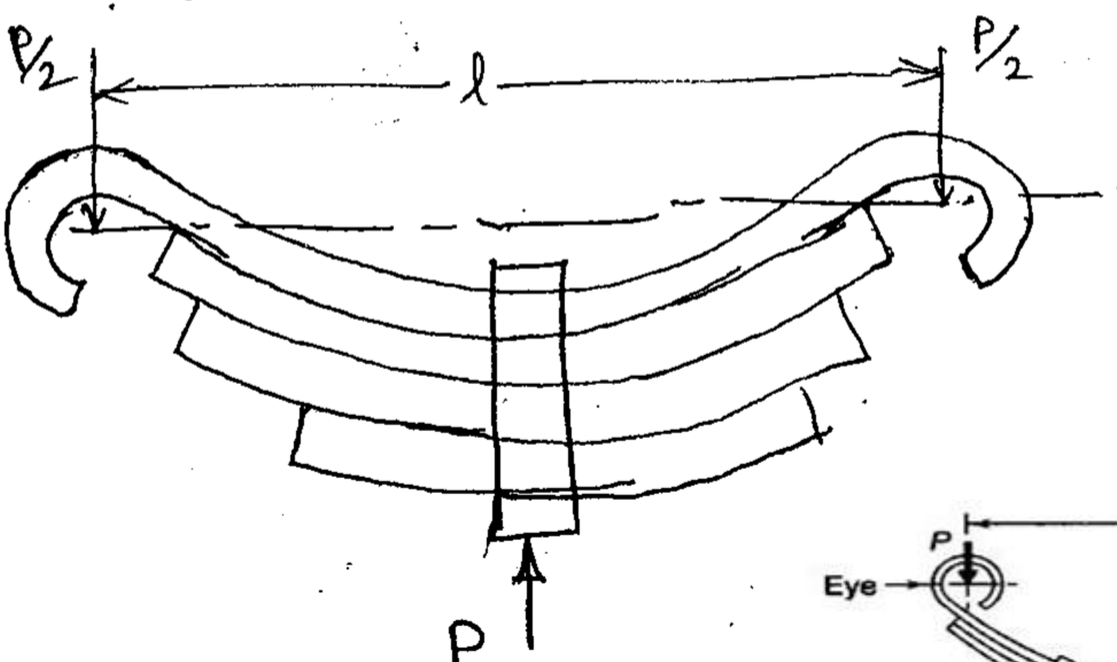
$$k = \frac{W}{\delta};\quad 20\times10^3 = \frac{300}{\delta};\quad \delta = 0.015\text{ m}$$

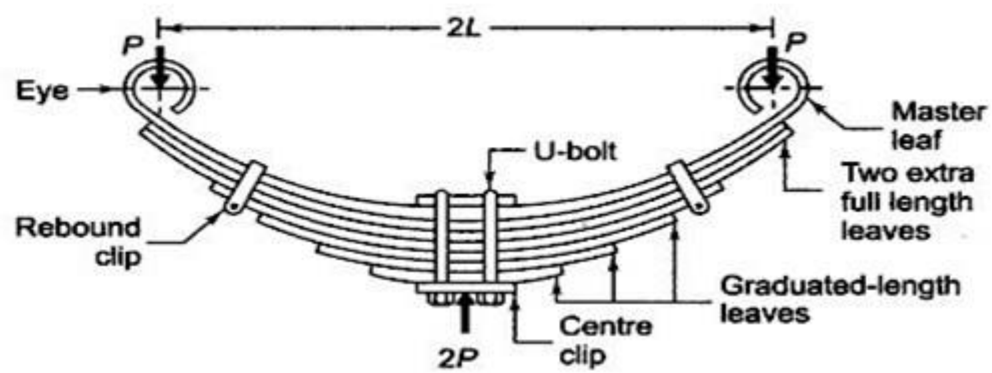
$$\delta = \frac{64WR^3n}{Cd^4};\quad 0.015 = \frac{64\times300\times(0.0127)^3\times n}{80\times10^9\times(0.006)^4}$$

$$n = 39.54\text{ Coils}$$

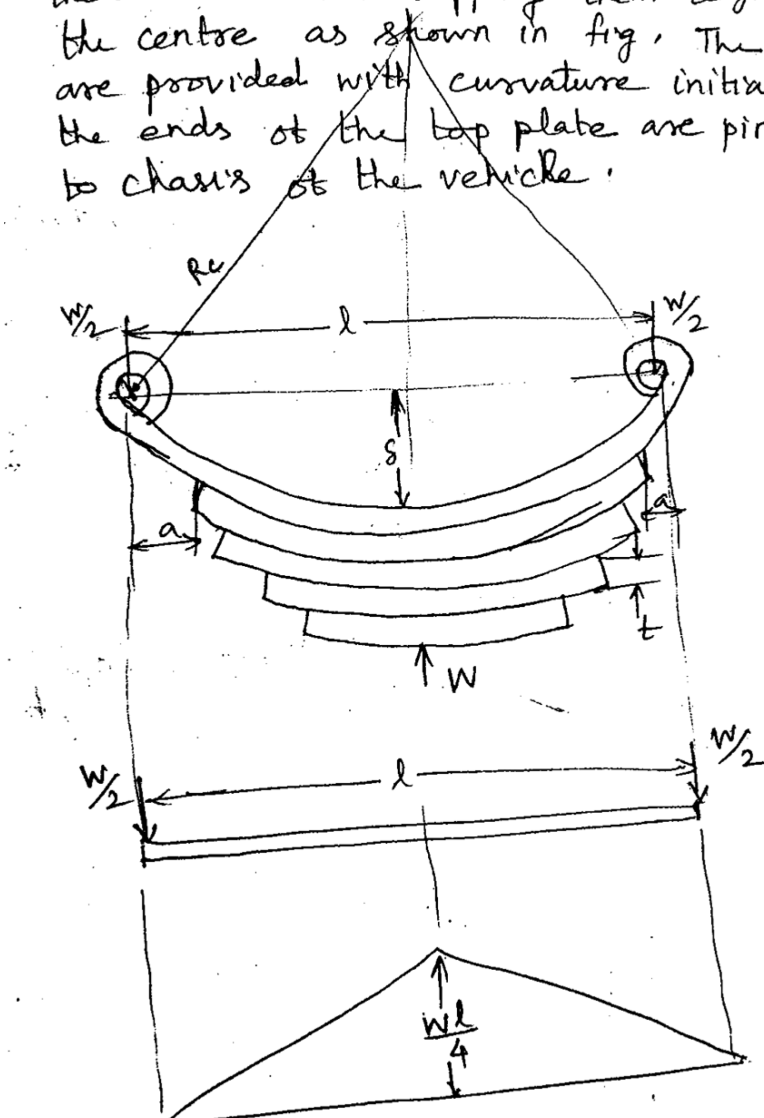
Laminated or leaf springs

These springs are called semielliptical, leaf or carriage springs and find their use in trucks, trains trolleys etc.





These springs are formed by placing a number of parallel flat metal strips one over the other and strapping them together at the centre as shown in fig. The plates are provided with curvature initially and the ends of the top plate are pin jointed to chassis of the vehicle.



B.M. Diagram.

Let l – Spring span length.
t – Thickness of each plate
b – width of each plate.
n – no. of plates
W – force acting on the spring.
f, σ_b – max. bending stress
δ – original central deflection.

The spring can be assumed as a simply supported beam.

$\therefore$ Bending moment at the centre $= M = \frac{Wl}{4}$ —— ①

moment of inertia of each plate $I = \frac{bt^3}{12}$

From theory of simple bending

$$\frac{M}{I} = \frac{f}{y} \quad \text{where } y = t/2$$

$$M = \frac{I}{y} \times f = \frac{bt^3}{12} \times \frac{f}{t/2} = \frac{fbt^2}{6}$$

Total resisting moment in the plates

$$M = \frac{nfbt^2}{6} \quad \text{—— ②}$$

Equating ① and ②

$$\frac{Wl}{4} = \frac{nfbt^2}{6}$$

$$\boxed{f = \frac{3Wl}{2nbt^2}}$$

(b) central deflection (δ)

The curvature of the spring is of true circular arc (from the geometry of a circle)

$$\therefore \delta = \frac{l^2}{8R_c} \quad \text{(where } R_c \text{ - radius of curvature)}$$

From theory of simple bending.

$$\frac{f}{y} = \frac{E}{R} \quad ; \therefore R_c = \frac{Ey}{f} = \frac{Et}{2f}$$

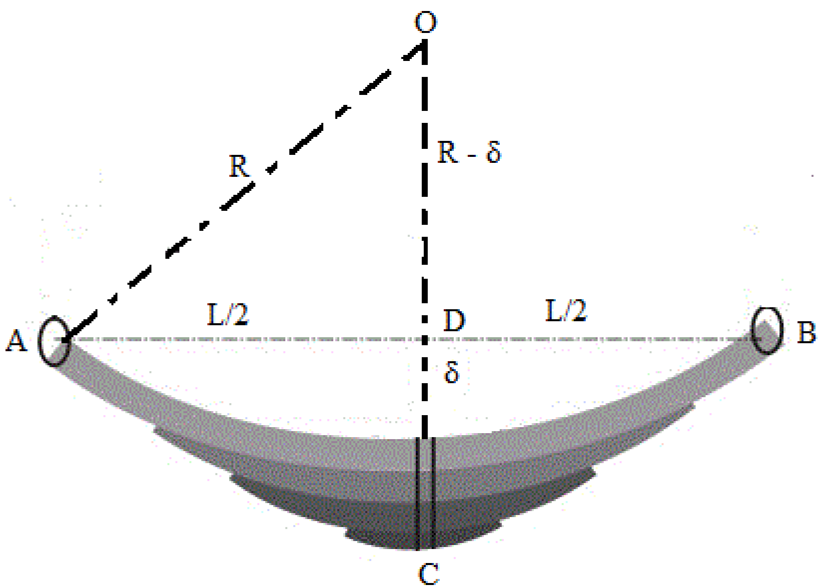
$$\therefore \delta = \frac{l^2}{8\frac{Et}{2f}}, \quad \boxed{\delta = \frac{fl^2}{4Et}} = \frac{\frac{3wl}{2nbt^2} \times l^2}{4Et}$$

$$\boxed{= \frac{3wl^3}{8Enbt^3}}$$

$\delta =$

Strain energy (U)

$$U = \frac{\sigma_b^2}{6E} \times \text{volume of spring.}$$



Let us consider here the triangle AOD;

$AO^2 = OD^2 + AD^2$

$R^2 = (R - \delta)^2 + (L/2)^2$

$R^2 = R^2 + \delta^2 - 2R.\ \delta + L^2/4$

$R^2 = R^2 - 2R.\ \delta + L^2/4$

We have neglected small term i.e. δ^2

$2R.\ \delta = L^2/4$

$\delta = L^2/8R$

(4) A carriage spring is to be 600 mm long and made of 9.5 mm thick steel plates and 50 mm broad. How many plates are required to carry a load of 4.5 KN, without the stress exceeding 230 MN/m^2.

What would be central deflection and would be the initial radius of curvature, if plates straighten under the load for plate.

$E = 200$ GN/m^2.

Soln: $l = 600$ mm $= 0.6$ m

$t = 9.5$ mm $= 0.0095$ m, $b = 50$ mm $= 0.050$

$W = 4.5$ KN, $E = 200$ GN/m^2, $\sigma_b = 230$ MN/m^2

No. of plates n

$$f = \frac{3Wl}{2nbt^2} \; ; \quad 230 \times 10^6 = \frac{3 \times 4.5 \times 10^3 \times 0.6}{2 \times n \times 0.05 \times (0.0095)^2}$$

$n = 3.9$ Say 4 (ANS)

Initial radius of curvature, R_c

$$\frac{f}{y} = \frac{E}{R_c} \; ; \quad R_c = \frac{E}{f} \times y = \frac{E}{f} \times \frac{t}{2}$$

$$= \frac{200 \times 10^9 \times 0.0095}{230 \times 10^6 \times 2}$$

$= 4.13$ m.

Central deflection δ

$$\delta = \frac{fl^2}{4Et} = \frac{230 \times 10^6 \times 0.6^2}{4 \times 200 \times 10^9 \times 0.0095}$$

$= \sout{0.0181 \text{ m}}$ 0.0108 m

(5) A carriage spring is built up of 9 plates of each 75 mm wide and 6 mm thick. Find the length of the spring if it carries a central load of 4 KN. The stress is limited to 150 N/mm². Find also the central deflection of the spring. $E = 0.2 \times 10^6$ N/mm²

Soln:

$$f = \frac{3Wl}{2nbt^2}$$

$$150 \times 10^6 = \frac{3 \times 4 \times 10^3 \times l}{2 \times 9 \times 0.075 \times (0.006)^2}$$

$$l = 0.6075 \text{ m} = 607.5 \text{ mm}$$

Central deflection $\delta = \frac{fl^2}{4Et} = \frac{150 \times 10^6 \times (0.6075)^2}{4 \times 0.2 \times 10^{12} \times 0.006}$

$$= 0.01153 \text{ m}$$

$$= 11.53 \text{ mm}$$

University question (Nov/Dec 2016)

13. (a) (i) A solid shaft has to transmit the Power 105 kW at 2000 r.p.m. The maximum torque transmitted in each revaluation exceeds the mean by 36%. Find the suitable diameter of the shaft, if the shear stress is not to exceed 75 N/mm² and maximum angle of twist is 1.5° in a length of 3.30 m and $G = 0.80 \times 10^5$ N/mm². (8)

(ii) A laminated spring carries a central load of 5200 N and it is made of 'n' number of plates, 80 mm wide. 7 mm thick and length 500 mm. Find the numbers of plates, if the maximum deflection is 10 mm. Let $E = 2.0 \times 10^5$ N/mm². (5)

Or

(b) (i) A stepped solid circular shaft is built in at its ends and subject to an externally applied torque T at the shoulder as shown in fig. Q.13(b)(i). Determine the angle of rotation θ of the shoulder section when T is applied. (7)

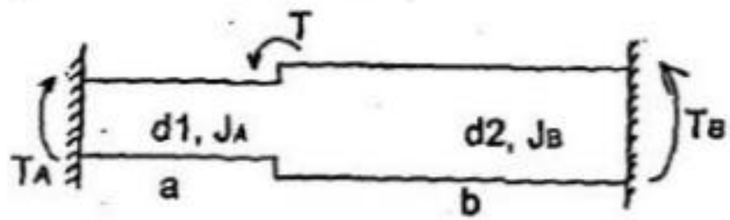


Fig. Q.13(b)(i)

(ii) A closed coiled helical spring is to be made out of 5 mm diameter wire 2 m long so that it deflects by 20 mm under an axial load of 50 N. Determine the mean diameter of the coil. Take $C = 8.1 \times 10^4$ N/mm². (6)

- A solid circular shaft transmits 75 kW power at 200 rpm. Calculate the shaft diameter, if the twist in the shaft is not to exceed 1° in 2 metres length of the shaft and shear stress is limited to 50 N/mm^2. Take C = 1 x 10^5 N/mm^2.
- A solid steel shaft transmits a power 20 kW at 60 rpm. Determine the smallest safe diameter of the shaft if the shear stress is not to exceed 40 N/m^2 and the angle of twist is limited to 5° in a length of 3 m. Take G = 80 GPa
- A hollow shaft of external diameter 120 mm transmits 300 kW power at 200 rpm. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm^2.
- A closely coiled helical spring made from round steel rod is required to carry a load of 1000 N for a maximum stress of 400 MN/m^2. The spring stiffness is 20 N/mm. The mean diameter of the helix is 100 mm and modulus of rigidity of the material is 80 GN/m^2. Calculate (1) diameter of the wire and (2) the number of turns required for the spring
- A laminated spring carries a central load of 5200 N and it is made of 'n' number of plates, 80 mm wide, 7 mm thick and length 500 mm. Find the number of plates, if the maximum deflection is 10 mm. Let E = 2.0 x 10^5 N/mm^2.

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Thank you

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STRENGTH OF MATERIALS FOR MECHANICAL ENGINEERS (CE8395)

UNIT-4

DEFLECTION OF BEAMS

by

Dr. B. Janarthanan

Professor

Department of Mechanical Engineering

Syllabus

- Double Integration method – Macaulay's method – Area moment method for computation of slopes and deflections in beams – Conjugate beam and strain energy – Maxwell's reciprocal theorems.

Course objective:

To compute slopes and deflections in determinate beams by various methods.

Course outcome:

- **After completion of this unit students should be able to:**

Calculate the slope and deflection in beams using different methods.

Introduction

- The cross section of a beam has to be designed in such a way that it is strong enough to limit the bending moment and shear force that are developed in the beam. This criterion is known as "**strength criterion**"
- Another criterion for beam design is that the maximum deflection must not exceed a certain limit and the beam must be stiff enough to resist the deflection caused due to loading. This criterion is known as "**stiffness criterion**"
- It is therefore necessary to predict the deflection of members under lateral or transverse loads

Definitions

(i)DEFLECTION :-

- The vertical distance in transverse direction between positions of axis before and after loading at the section of the beam, is defined as the deflection of beam at that section.

(ii) ELASTIC CURVE(OR, DEFLECTION CURVE):-

- The neutral axis in its deflected position after loading of the beam is known as its elastic curve or deflection curve

(iii) SLOPE:-

- The slope of the beam at any section is defined as the angle (in radians) of inclination of the tangent drawn at that section to the axis in its deflected position after loading, measured w. r. t. the undeformed axis.

(iv) FLEXURAL RIGIDITY(EI):-

- The product of modulus of elasticity and Moment of Inertia is known as Flexural rigidity.

DIFFERENTIAL EQUATION OF ELASTIC CURVE

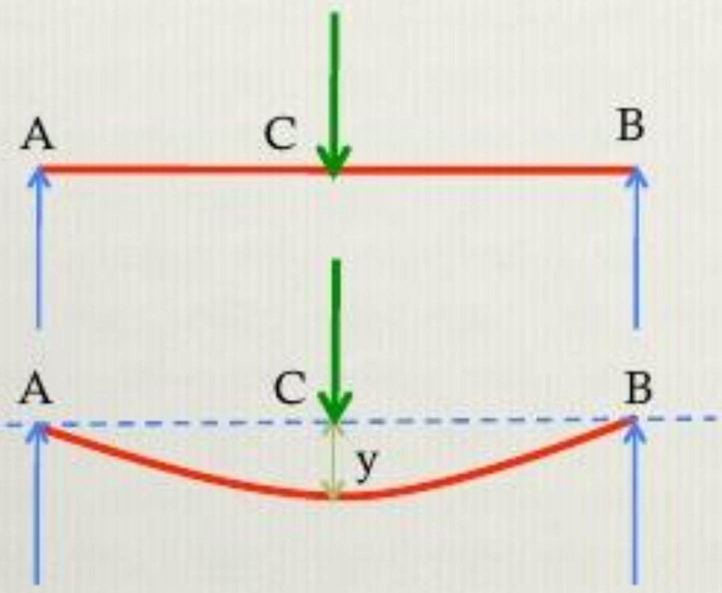
- Radius of curvature

$$R = \left| \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} \right|$$

- $\frac{1}{R} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$
- $\frac{M}{I} = \frac{E}{R}$

- $M = EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ → differential eqn. of flexure

Relationship



Deflection $= y$

Slope $= \dfrac{dy}{dx}$

Bending moment $= EI\dfrac{d^2y}{dx^2}$

Shearing force $= EI\dfrac{d^3y}{dx^3}$

Rate of loading $= EI\dfrac{d^4y}{dx^4}$

ASSUMPTIONS MADE IN THE DEFLECTION:-

(i) Axis of the beam is horizontal before loading.

(ii) Deflection due to S.F. is negligible. It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.

(iii) (a) Simple Bending equation $M/I=\sigma/y=E/R$ is applicable and all the assumptions made in simple bending theory are valid.

(b) Material of the beam is homogenous, isotropic and obey Hook's law ..

(c) The modulus of elasticity is same in compression as well as in tension.

(d) Plane section remain plane before and after bending

Methods

- Methods for finding slope and deflection of beams:

(i) Double integration method / Direct integration

(ii) Macaulay's method

(iii) Moment area method

(iv) Conjugate beam method

Double integration method

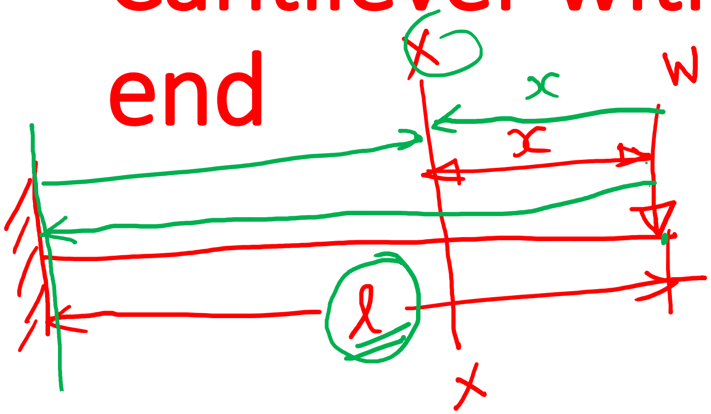
- The beam differential equation is integrated twice – deflection of beam at any c/s.

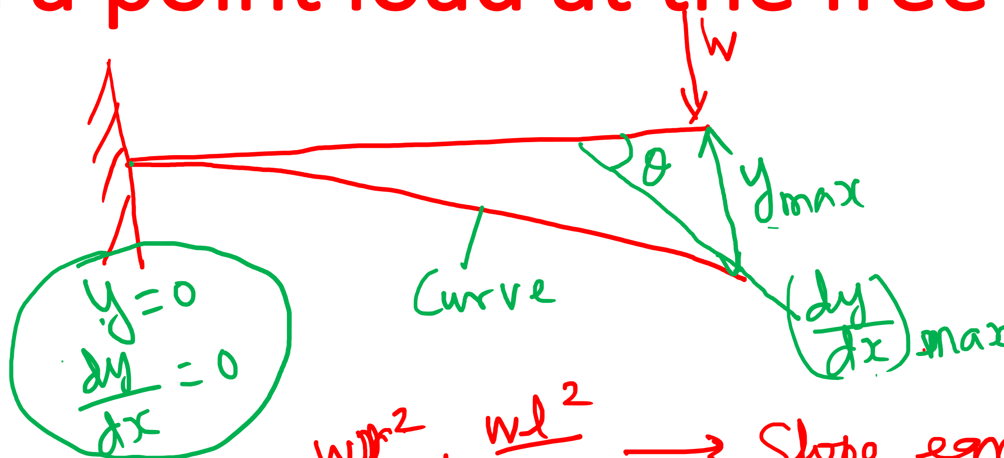
$$EI\frac{dy}{dx} = \int M \cdot dx + C_1$$ from which slope can be calculated

$$EI \cdot y = \iint (M \cdot dx) + C_1 x + C_2$$ from which deflection is known

Where C1 and C2 are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Cantilever with a point load at the free end





$M_x = -Wx$

$EI\frac{d^2y}{dx^2} = M$

$EI\frac{d^2y}{dx^2} = -Wx$ —— (1)

Integrating w.r.to x

$EI\left(\frac{dy}{dx}\right) = -W\frac{x^2}{2} + C_1$

at $x = l$, $\frac{dy}{dx} = 0$

$0 = -\frac{Wl^2}{2} + C_1 \rightarrow C_1 = \frac{Wl^2}{2}$

$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{Wl^2}{2}$ → Slope eqn (2)

$x = 0 \rightarrow \left(\frac{dy}{dx}\right)_{max}$

$EI\left(\frac{dy}{dx}\right)_{max} = \frac{Wl^2}{2}$

$$\left(\frac{dy}{dx}\right)_{max} = \frac{Wl^2}{2EI}$$

Integrating eqn (2)

$EI\,y = -\frac{Wx^3}{6} + \frac{Wl^2}{2}x + C_2$

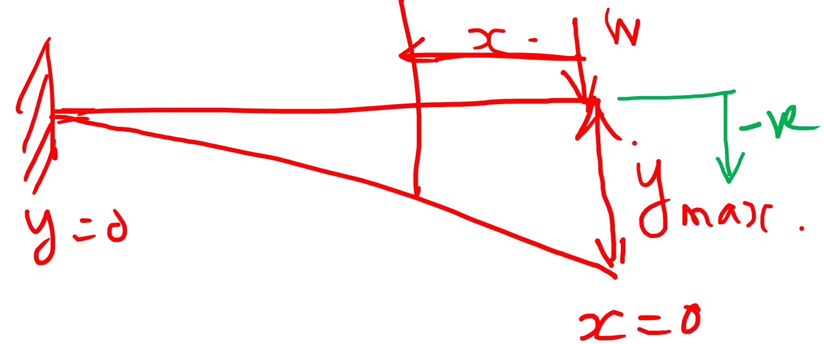
$$EIy = \frac{-Wx^3}{6} + \frac{Wl^2}{2}x + C_2$$

at $x = l$, $y = 0$.

$$EI(0) = \frac{-Wl^3}{6} + \frac{Wl^3}{2} + C_2$$

$$C_2 = \frac{Wl^3}{6} - \frac{Wl^3}{2} = \frac{Wl^3 - 3Wl^3}{6}$$

$$C_2 = \frac{-2Wl^3}{6} = -\frac{Wl^3}{3}.$$



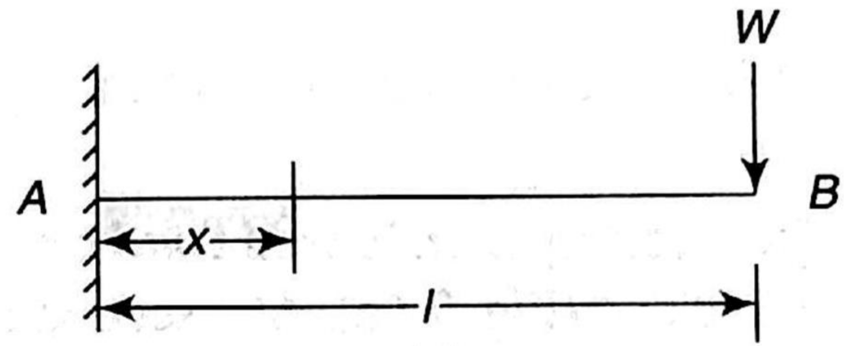
$$EIy = -\frac{Wx^3}{6} + \frac{Wl^2}{2}x - \frac{Wl^3}{3}$$ (deflection eqn.)

at $x = 0$, $y = y_{max}$.

$$EIy_{max} = 0 + 0 - \frac{Wl^3}{3}$$

$$y_{max} = \frac{-Wl^3}{3EI}$$

Cantilever with a point load at the free end



Bending Moment at the section =− W(l−x),

Or EI $\frac{d^2y}{dx^2}$ = − W(l−x)

Integrating, EI $\frac{dy}{dx} = -W\left(lx - \frac{x^2}{2}\right) + C_1$

At x = 0, $\frac{dy}{dx} = 0$, therefore $C_1 = 0$,

Thus EI $\frac{dy}{dx} = -W\left(lx - \frac{x2}{2}\right)$

Or ***Slope***, $\frac{dy}{dx} = -\frac{W}{2EI}(2lx - x^2)$

Cantilever with a point load at the free end

$$EI\frac{dy}{dx} = -W\left(lx - \frac{x2}{2}\right)$$

Integrating again, $EI\,y = -W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2$

At x = 0, y = 0, therefore $C_2 = 0$,

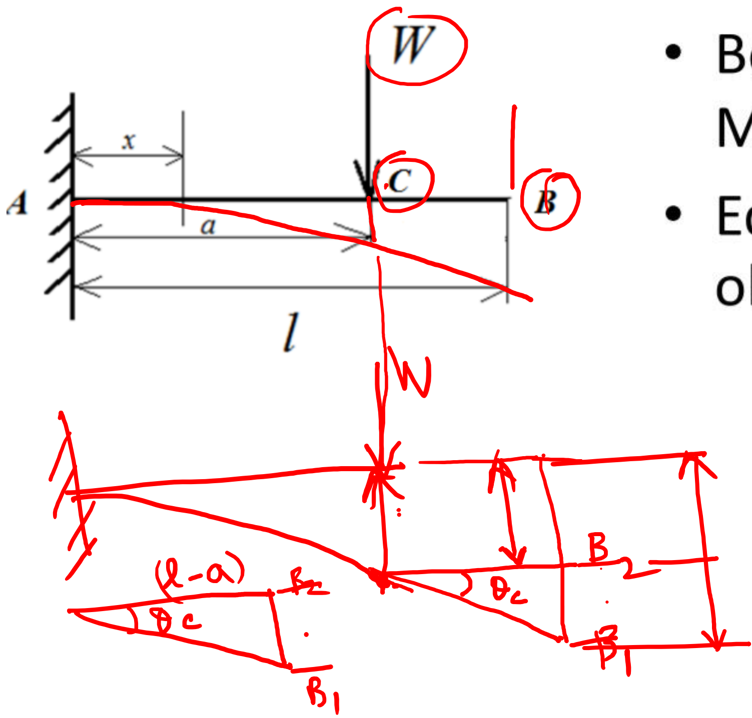
Thus EIy = -W $\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$

Or ***Deflection, y =*** $-\frac{W}{6EI}\left(3lx^2 - x^3\right)$

At the free end, x = l, the slope and deflection are maximum and are given by

Slope = $-\frac{Wl^2}{2EI}$ *and deflection* = $-\frac{Wl^3}{3EI}$

Concentrated load not at free end



- Between A and C at any distance x from A, M = -W (a-x)
- Equations of slope and deflection can be obtained as in previous case (replacing ℓ by a)

$$\textit{Slope,}\quad \frac{dy}{dx} = -\frac{W}{2EI}(2ax - x^2)$$

$$\textit{Deflection,}\ y = -\frac{W}{6EI}(3ax^2 - x^3)$$

At C, x=a; hence $\frac{dy}{dx} = -\frac{Wa^2}{2EI}$

and $y = -\frac{Wa^3}{3EI}$

$$\theta_B = \theta_C = -\frac{Wa^2}{2EI}$$

$$y_B = y_C + B_2B_1$$

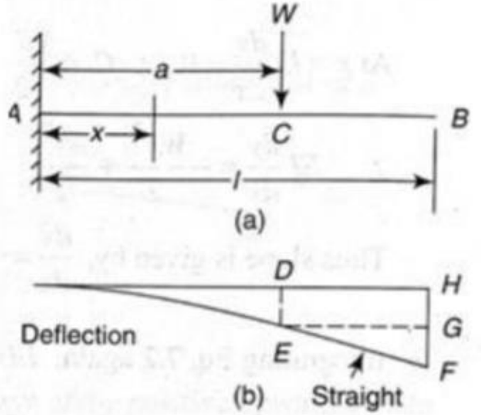
$$\tan\theta_C = \frac{B_2B_1}{l-a}$$

$$B_2B_1 = (l-a)\theta_C = (l-a)\frac{-Wa^2}{2EI}$$

Concentrated load not at free end

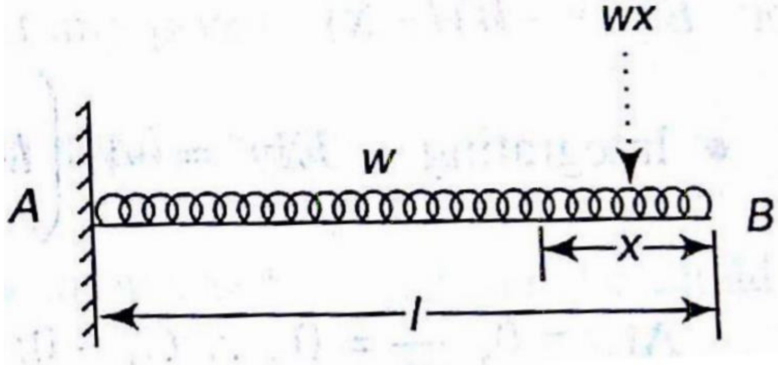
- The beam will bend only between A and C and between B and C it will remain straight (as BM between B and C = 0)
- Hence slope at B = slope at C = dy/dx = GF/GE = $-\frac{Wa^2}{2EI}$
- Now deflection at B = deflection at C + GF
- = deflection at C +($-\frac{Wa^2}{2EI}$)GE

ie, **Deflection at B** $= -\frac{Wa^3}{3EI} - \frac{Wa^2}{2EI}(l-a)$



If W is at the midpoint, deflection $= \left[\frac{W(l/2)^3}{3EI} + \frac{W(l/2)^2}{2EI} \cdot \frac{l}{2}\right] = \frac{5Wl^3}{48EI}$

Cantilever with udl throughout the length



At a section at a distance x from the free end,

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

Integrating, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$

At $x = l$, $\frac{dy}{dx} = 0$, $\therefore\ C_1 = \frac{wl^3}{6}$

Thus, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6} = \frac{w}{6}(l^3 - x^3)$

Cantilever with udl throughout the length

Integrating again, $EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x + C_2$

At A, $x = l, y = 0$, $\therefore \; 0 = -\frac{wl^4}{24} + \frac{wl^3}{6} \cdot l + C_2$

or $C_2 = -\frac{wl^4}{8}$

Thus, $EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x - \frac{wl^4}{8}$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = \frac{w}{6EI}(l^3 - x^3) \text{ and } y = -\frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$$

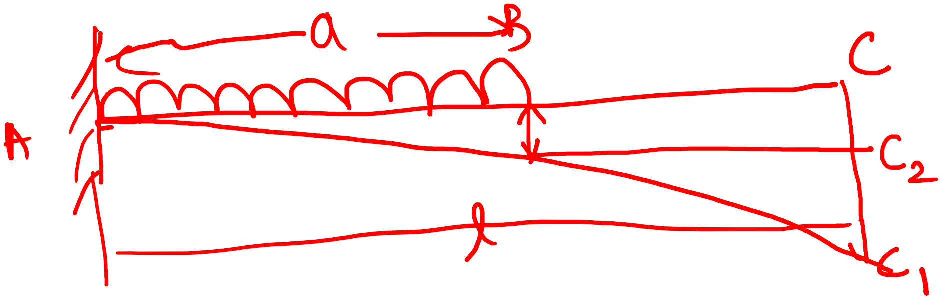
Cantilever with udl throughout the length

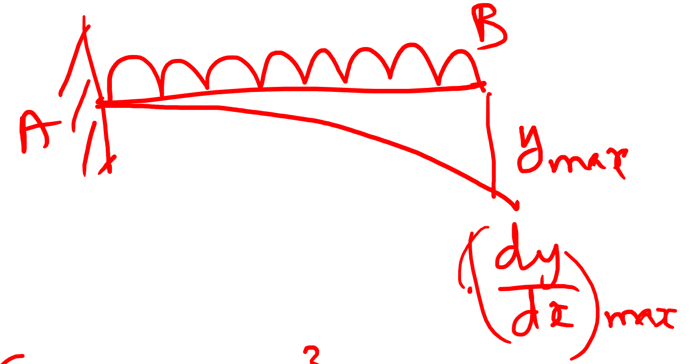
$$\text{Maximum slope} = \frac{wl^3}{6EI} \text{ at } x = 0$$

$$\text{Maximum deflection} = -\frac{wl^4}{8EI} \text{ at } x = 0$$

If origin is taken at the fixed end, slope and deflection can be worked out to be

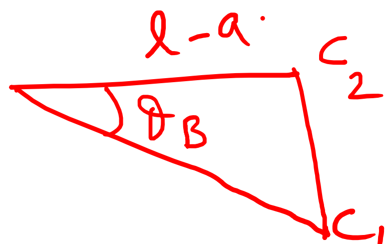
$$y' = -\frac{w}{6EI}(3l^2x - 3lx^2 + x^3); \quad y = -\frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4)$$





Slope at B = Slope at C

$$\theta_C = \frac{wa^3}{6EI}$$



$$\theta_B = \frac{wa^3}{6EI}$$

$$\tan\theta_B = \frac{C_2C_1}{l-a}$$

$$y_B = \frac{wa^4}{8EI}$$

$$y_C = y_B + C_2C_1$$

$$= \frac{wa^4}{8EI} + (l-a)\frac{wa^3}{6EI}$$

$$C_2C_1 = (l-a)\,\theta_B$$

$$= (l-a)\frac{wa^3}{6EI}$$

2 marks questions

15) In a support beam of 3m span carrying uniformly distribution load throughout the length the slope at the support is 1°. What is the max deflection in the beam? (Apr/May 2019)

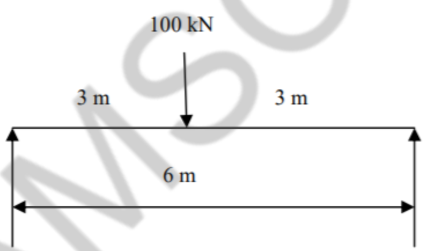
$$\theta_A = \frac{\omega \ell^3}{24EI} = 1° = \frac{\pi}{180°}$$

Max deflection (y_{max}) = $\frac{5}{384}\frac{\omega \ell^4}{EI}$

$$= \frac{\omega \ell^3}{24EI} \times \frac{5\ell}{16} = \frac{\pi}{180°} \times \frac{5 \times 3}{16}$$

$$\boxed{y_{max} = 0.0164}$$

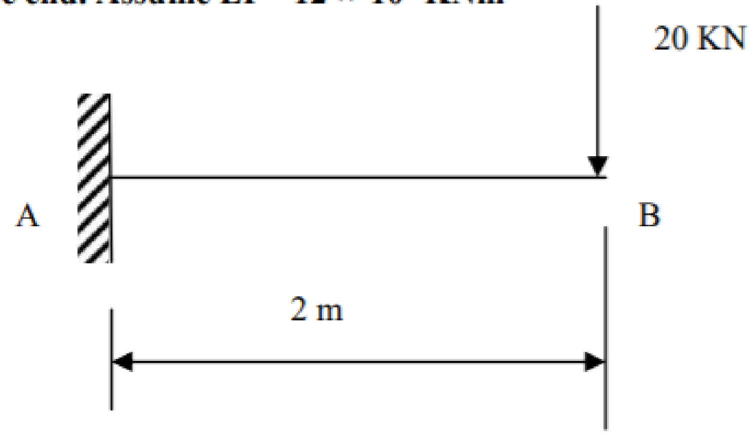
16) Calculate the maximum deflection of a simply support beam carrying a point load of 100 KN at mid span. Span = 6m; EI = 20,000 KN/m²



$$y_{max} = \frac{\omega \ell^3}{48EI} = \frac{100 \times 6^3}{48 \times 20000} = 0.0225\,m$$

$$\boxed{y_{max} = 22.5 \text{ mm}}$$

17) A cantilever beam of span 2m is carrying a point load of 20 KN in the free end. Calculate the slope at the free end. Assume EI = 12 × 10^3 KNm^2



$$\theta_B = \frac{\omega \ell^2}{2EI}$$

$$= \frac{20 \times 2^2}{2 \times 12 \times 10^3}$$

$$\boxed{\theta_B = 0.0033 \text{ rad}}$$

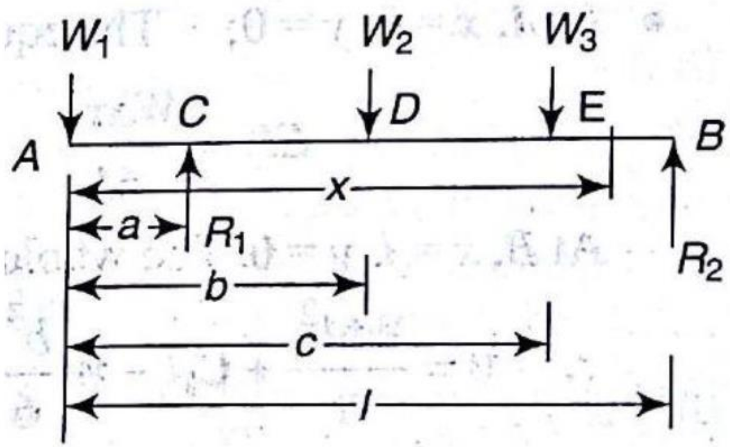
Macaulay's method

While applying the double integration method, a separate expression for the bending moment is needed to be written for each section of the beam, each producing a different equation with its own constants of integration.

The method is convenient for simple cases

In Macaulay's method, a single equation is written for the bending moment for all the portions of the beam. The equation is formed in such a way that the same constants of integration are applicable to all portions.

Macaulay's method



$$EI\frac{d^2y}{dx^2} = M = -W_1x \mid + R_1(x-a) \mid - W_2(x-b) \mid - W_3(x-c)$$

In the above expression, there are separation lines.

- The portion to the left of the first separation line is valid for the portion *AC*.
- The portion to the left of the second separation line is valid for the portion *CD*.
- The portion to the left of the third separation line is valid for the portion *DE*.
- The whole of the expression is valid for the portion *EB*.

Macaulay's method

It may be noted that the same expression is applicable to all the portions of the beam if all negative terms inside the brackets are omitted for a particular section. If x is less than c, then the last term is omitted. If x is less than b, then the last two terms are omitted and so on. While integrating, the brackets are integrated as a whole, i.e.,

$$EI\frac{d^2y}{dx^2} = M = -W_1x \mid + R_1(x-a) \mid - W_2(x-b) \mid - W_3(x-c)$$

$$EI\frac{dy}{dx} = -W_1\frac{x^2}{2} + C_1 \Big| + \frac{R_1}{2}(x-a)^2 \Big| - \frac{W_2}{2}(x-b)^2 \Big| - \frac{W_3}{2}(x-c)^2$$

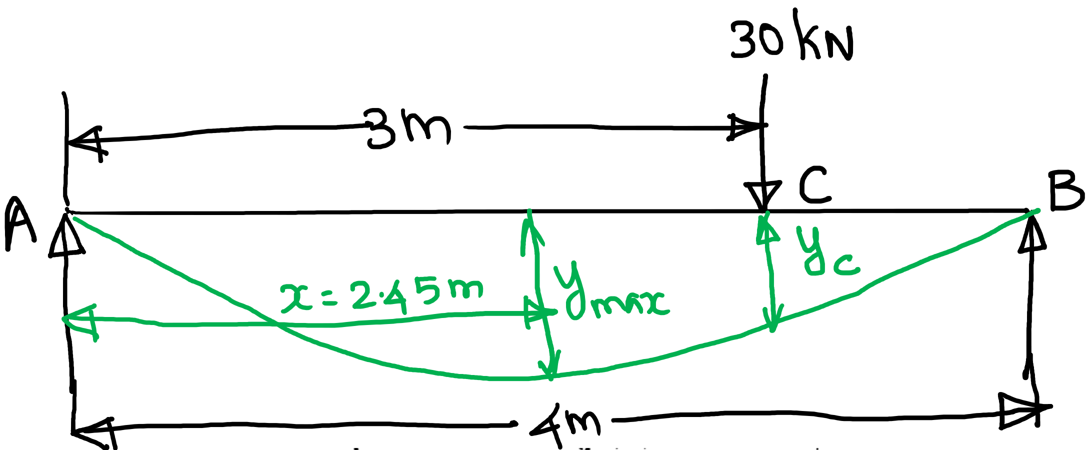
$$EIy = -W_1\frac{x^3}{6} + C_1x + C_2 \Big| + \frac{R_1}{6}(x-a)^3 \Big| - \frac{W_2}{6}(x-b)^3 \Big| - \frac{W_3}{6}(x-c)^3$$

Problem

A beam with a span of 4.5m carries a point load of 30kN at 3m from left support. $I_{xx} = 54.97 \times 10^{-6}\ m^4$ and $E = 200 GN/m^2$, Find,

(i) Deflection under the load

(ii) Position and amount of Maximum deflection.



To find Reactions

Take moments about A

$R_B \times 4.5 = 30 \times 3 = 90$

$R_B = 20\ kN$

$R_A = 10\ kN\ (30-20)$

The B.M. at any section X distant x from A

$$M_x = EI\frac{d^2y}{dx^2} = 10x \,|\, - 30(x-a)$$

Integrating

$$EI\frac{dy}{dx} = \frac{10x^2}{2} + C_1 \Big| - \frac{30(x-3)^2}{2}$$

Integrating again

$$EIy = \frac{10x^3}{6} + C_1x + C_2 \Big| - \frac{30(x-3)^3}{6}$$

At A deflection is Zero $x=0,\ y=0$, $\therefore C_2 = 0$.

At B " $x=l,\ y=0$

$$0 = \frac{10l^3}{6} + C_1 l - \frac{30(4.5-3)^3}{6}$$

$$0 = \frac{10\times 4.5^3}{6} + C_1\times 4.5 - \frac{30(1.5)^3}{6}$$

$$C_1 = -30.$$

$$EIy = \frac{5}{3}x^3 - 30x \Big| - 5(x-3)^3$$ deflection eqn.

deflection at C, $x=3$

$$EIy_c = \frac{5}{3}(3)^3 - 30(3) \Big| - 5(3-3)^3$$

$$y_c = \frac{-45}{EI} = \frac{-45}{200\times10^6\times 54.97\times10^{-6}}$$

$= -0.00409$ m

$= -4.09$ mm.

Slope eqn.

$$EI\frac{dy}{dx} = 5x^2 - 30 \Big| - 15(x-3)^2$$

The max. deflection will occur, on the larger segment AC. The slope is Zero at the point of maximum deflection

$$0 = 5x^2 - 30 - 15(x-3)^2$$

$$0 = 5x^2 - 30 - 15x^2 + 15\times6x - 15\times9$$

$$0 = -10x^2 + 90x - 165$$

$$2x^2 - 18x + 33 = 0.$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{18\pm\sqrt{18^2-4\times2\times33}}{2\times2}$$

$= 2.56$ m and 6.43 m

($x = 2.49$ m, $y_{max} = 4.456$)

Max. deflection put $x = 2.56$ in deflection eqn.

$$EIy = \frac{5}{3}x^3 - 30x - 5(x-3)^3$$

$$EIy_{max} = \frac{5}{3}(2.56)^3 - 30(2.56) = -48.83$$

$$y_{max} = \frac{-48.83}{EI} = \frac{-48.83}{200\times10^6\times 54.97\times10^{-6}}$$

$= 0.00444$ m

$= 4.44$ mm

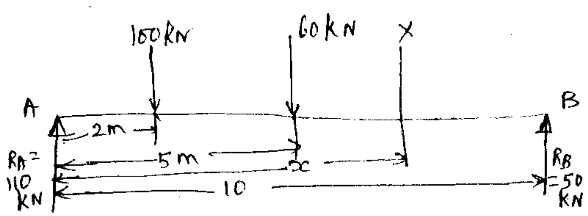
Problem

- A beam is simply supported at its ends over a span of 10 m and carries two concentrated loads of 100 kN and 60 kN at a distance of 2m and 5m respectively from the left support. Calculate

I. Slope at the left support

II. Slope and deflection under 100 kN load.

Assume EI = 36x10^4 kN-m^2



$R_A + R_B = 100 + 60 = 160$ kN

$R_B \times 10 = 60 \times 5 + 100 \times 2$

$R_B = 50$ kN , $R_A = 110$ kN

Taking Moment at x

$$EI\frac{d^2y}{dx^2} = R_A x \;\Big|\; -100(x-2) \;\Big|\; -60(x-5)$$

$$EI\frac{d^2y}{dx^2} = 110x \;\Big|\; -100(x-2) \;\Big|\; -60(x-5)$$

Integrating.

$$EI\frac{dy}{dx} = \frac{110x^2}{2} + C_1 \;\Big|\; -\frac{100(x-2)^2}{2} \;\Big|\; -\frac{60(x-5)^2}{2}$$

$$EI\frac{dy}{dx} = 55x^2 + C_1 \;\Big|\; -50(x-2)^2 \;\Big|\; -30(x-5)^2$$

Integrating again

$$EIy = \frac{55x^3}{3} + C_1x + C_2 \;\Big|\; -\frac{50(x-2)^3}{3} \;\Big|\; -\frac{30(x-5)^3}{3}$$

At $x=0$, $y=0 \Rightarrow C_2 = 0$.

At $x=10$, $y=0 \Rightarrow C_1 = -855$.

Slope eqn.

$$EI\frac{dy}{dx} = 55x^2 - 855 \;\Big|\; -50(x-2)^2 \;\Big|\; -30(x-5)^2$$

i) Slope at left support.

i.e, at A, $x = 0$.

$$36 \times 10^4 \times \frac{dy}{dx} = 55(0)^2 - 855$$

$$\frac{dy}{dx} = \frac{-855}{36 \times 10^4} = -2.375 \times 10^{-3} \text{ rad.}$$

ii) Slope at 100 kN load.

$x = 2$.

$$36 \times 10^4 \frac{dy}{dx} = 55(2)^2 - 855 - 50(2-2)^2$$

$$36 \times 10^4 \frac{dy}{dx} = -635$$

$$\frac{dy}{dx} = -1.76 \times 10^{-3} \text{ rad.}$$

Deflection at 100 kN load

$$EIy = \frac{55x^3}{3} - 855x + 0 \;\Big|\; - \frac{50(x-2)^3}{3} \;\Big|\; - \frac{30(x-5)^3}{3}$$

at $x = 2$.

$$36 \times 10^4 \times y = \frac{55(2)^3}{3} - 855(2) - \frac{50(2-2)^3}{3}$$

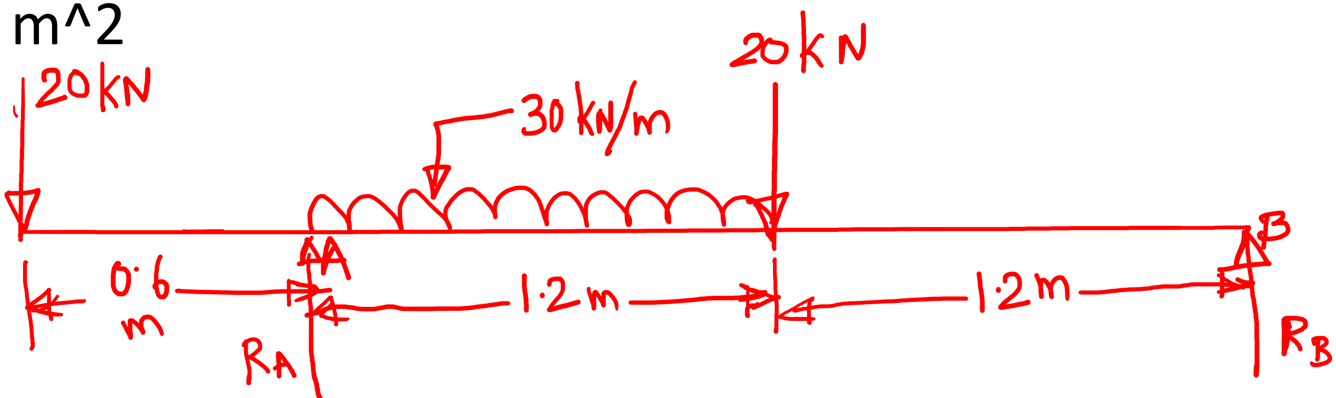
$$36 \times 10^4 \, y = -1563.33 \text{ kNm.}$$

$$y = -0.00434 \text{ m.}$$

$$= -4.34 \text{ mm.}$$

Problem

- In a beam shown in Fig. determine the slope at the left end C and deflection at 1 m from the left end. Take EI = 0.65 MN-m^2



x

$$R_B \times 2.4 = 20\times1.2 + 30\times1.2\times0.6 - 20\times0.6$$
$$= 24 + 21.6 - 12$$
$$R_B = 14\ kN$$
$$R_A = (30\times1.2) + 20 + 20 - 14 = 62\ kN$$

$$EI\frac{d^2y}{dx^2} = -20x \Big| + 62(x-0.6) \Big| - \frac{30(x-0.6)^2}{2} \Big| - 20(x-1.8) \Big| + \frac{30(x-1.8)^2}{2}$$

$R_B \times 2.4 = 20 \times 1.2 + 30 \times 1.2 \times 0.6 - 20 \times 0.6$

$= 24 + 21.6 - 12$

$R_B = 14\,kN$.

$R_A = (30 \times 1.2) + 20 + 20 - 14 = 62\,kN$

$$EI\frac{d^2y}{dx^2} = -20x + 62(x-0.6) - \frac{30(x-0.6)^2}{2}$$

$$- 20(x-1.8) + \frac{30(x-1.8)^2}{2}$$

$$EI\frac{dy}{dx} = \frac{-20x^2}{2} + C_1 + \frac{62(x-0.6)^2}{2} - \frac{15(x-0.6)^3}{3}$$

$$- \frac{20(x-1.8)^2}{2} + \frac{15(x-1.8)^3}{3}$$

$$EIy = \frac{-10x^3}{3} + C_1x + C_2 + \frac{31(x-0.6)^3}{3} - \frac{5(x-0.6)^4}{4}$$

$$- \frac{10(x-1.8)^3}{3} + \frac{5(x-1.8)^4}{4}$$

@ $x = 0.6\,m$, $y = 0$ and

@ $x = 3\,m$, $y = 0$.

$C_1 = -3.72$, $C_2 = 2.952$

$EI\frac{dy}{dx}(@\,x=0) = -3.72$

$\left(\frac{dy}{dx}\right)_c = \frac{-3.72}{0.65 \times 10^3} = -0.00572$ rad.

$EIy(@\,x = 1m) = -3.472$

$y(@\,x = 1m) = \frac{-3.472}{0.65 \times 10^3}$

$= -0.00534$ m.

- A simply supported beam of 8 m length carries two point loads of 64 kN and 48 kN at 1 m and 4 m from the left hand end. Find the deflection under each load and max. deflection E = 210 GPa and I = 180x10^6 mm^4

Moment area method

- Convenient for beams acted upon with point loads where BMD consists of triangles and rectangles.
- For the case of UDL, Macaulay's method is most uitable.

Mohr’s first Moment area method

- From the above **Mohr’s first moment-area theorem** can be stated as below:
- “The difference of slopes between any two points on an elastic curve of a beam is equal to the net area of the BMD between these points divided by EI”.

Mohr’s second Moment area method

- The above equation leads to the statement of Mohr’s second theorem.
- “The intercepts on a given line between the tangents to the elastic curve of a beam at any two points is equal to the net moment taken about the line of the area of the BMD between the two points divided by EI”.

Moment area theorems

- **1st - Theorem :**
- **Gives Slope of a Beam** and notation of slope by letter i (or) θ

$$\text{Slope} = \theta = \frac{\text{Area of Bending moment diagram (A)}}{EI}$$

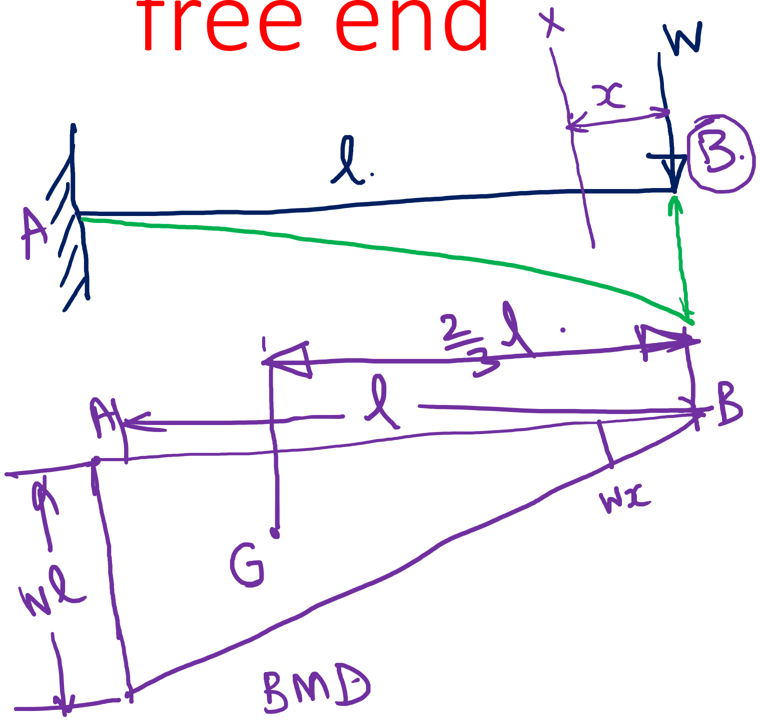
- Where EI is called Flexural Rigidity
- E = Young's Modulus of the material,
- I = Moment of Inertia of the beam.
- Slope is expressed in radians.

- **2nd – Theorem :**
- **Gives Deflection of a Beam** and notation with letter Y or Δ

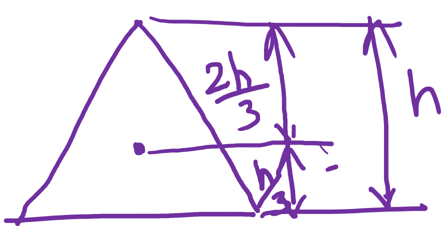
$$Y = \frac{\text{Area of BMD (A) x Centeroidal distance (x)}}{EI}$$

- Expressed in M, CM, MM

Cantilever carrying a point load at free end



$$\frac{1}{2} l \times Wl = \frac{1}{2} Wl^2$$



Let θ_B be the slope at B.

Slope

$$\theta_B = \frac{\text{Area of BMD between A and B}}{EI}$$

$$= \frac{\frac{1}{2} Wl^2}{EI} = \frac{1}{2} \frac{Wl^2}{EI} = \frac{Wl^2}{2EI}$$

Deflection at B (y_B)

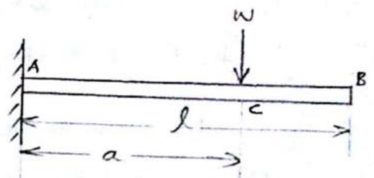
$$y_B = \frac{\text{Area of BMD} \times \text{Dist.}}{EI}$$

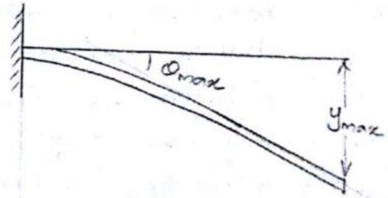
$$= \frac{\frac{1}{2} Wl^2 \times \frac{2}{3} l}{EI} = \frac{Wl^3}{3EI} \;//.$$

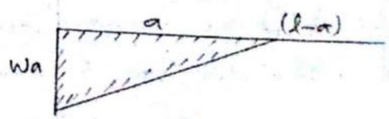
Cantilever carrying a concentrated load at any point

② . Concentrated load at any point.

Fig: shows a cantilever with a concentrated load ~~from~~ ~~at~~ acting at ~~c~~ at a distance 'a' from fixed end.







$$\theta_{max} = \frac{A}{EI}$$

Here, $A = \frac{1}{2} \times a \times Wa$

$$= \frac{Wa^2}{2}$$

$$\therefore \theta_{max} = \frac{Wa^2}{2EI}$$

Now $y_{max} = \frac{A\bar{x}}{EI}$

Here, $\bar{x} = (l-a) + \frac{2}{3}a.$

$$\therefore y_{max} = \frac{\frac{1}{2}Wa^2\left[(l-a) + \frac{2}{3}a\right]}{EI}$$

$$y_{max} = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(l-a)$$

At the point of application of the load, $y = \frac{\frac{1}{2}Wa^2 \times \frac{2a}{3}}{EI}$

$$y = \frac{Wa^3}{3EI}$$

By transfering the ref line to the point of application of load.

Cantilever carrying a udl

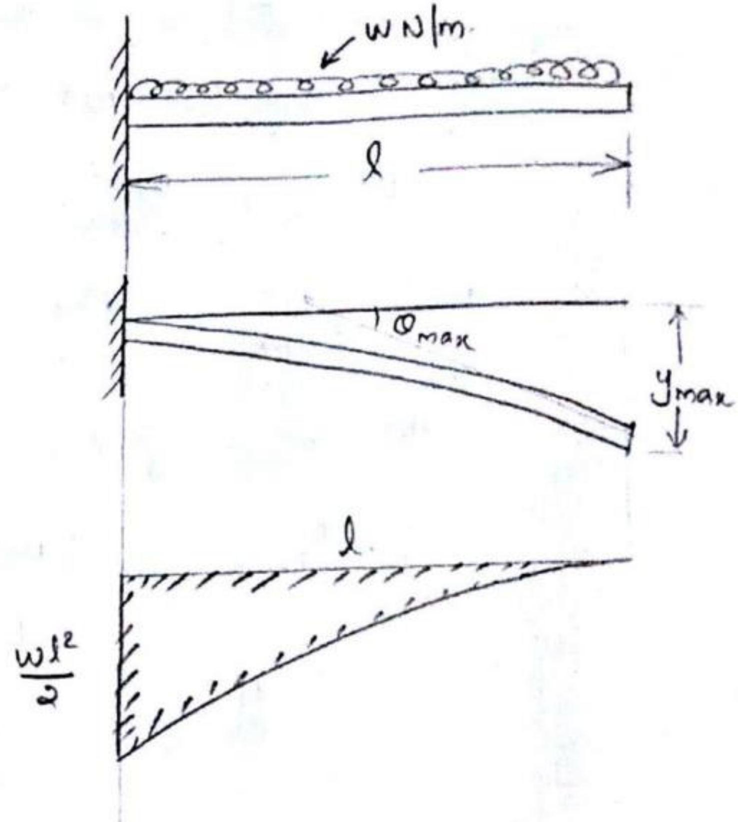
Cantilever beam with UD load over entire span.

We have $\theta_{max} = \dfrac{A}{EI}$.

Here area of BM diagram $A = \frac{1}{3} bh$

$$= \frac{1}{3} \times l \times \frac{wl^2}{2}$$

$$\therefore \theta_{max} = \frac{wl^3}{6EI}.$$



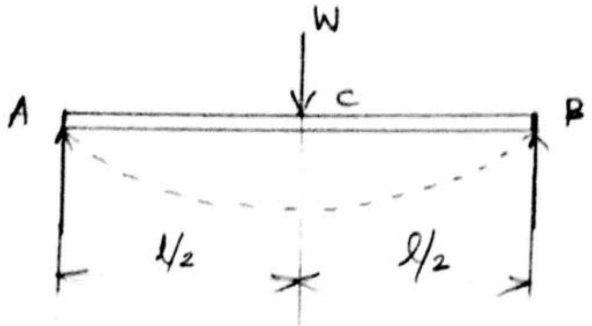
$$y_{max} = \frac{A\bar{x}}{EI}.$$

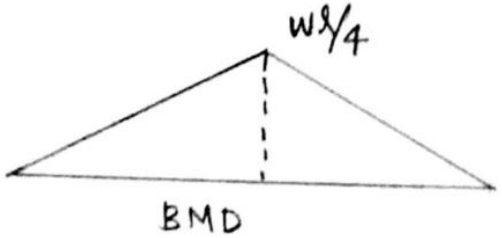
$$A = \frac{1}{3} \frac{wl^3}{2}.$$

$$\bar{x} = \frac{3}{4} l.$$

$$\therefore y_{max} = \frac{wl^4}{8EI}$$

Simply supported beam with a point load at mid span





Area of the BMD between A and C $= \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{Wl}{4}$

$= \frac{Wl^2}{16}$

We have $\theta_{max} = \frac{A}{EI}$

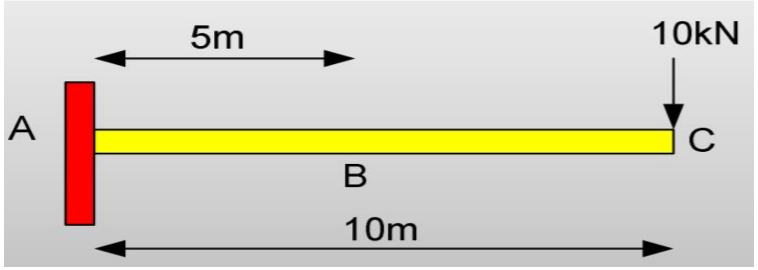
$\therefore$ Slope at A $= \frac{Wl^2}{16EI}$

and $y_{max} = \frac{A\bar{x}}{EI} = \frac{\left(\frac{1}{2} \cdot \frac{Wl}{4} \cdot \frac{l}{2}\right)\left(\frac{2}{3} \frac{l}{2}\right)}{EI}$

ie, $y_{max} = \frac{Wl^3}{48EI}$

DETERMINE THE SLOPE AT POINTS B AND C OF THE BEAM SHOWN BELOW. TAKE E = 200 GPa AND
$I = 360 \times 10^6\ mm^4$

$$\theta_B = \theta_{B/A} = -\left(\frac{50kNm}{EI}\right)(5m) - \frac{1}{2}\left(\frac{100kNm}{EI} - \frac{50kNm}{EI}\right)(5m)$$
$$= -\frac{375kNm^2}{EI}$$



$$-\frac{375kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00521rad$$

The –ve sign indicates that the angle is measured clockwise from A, Fig 8.15(c)

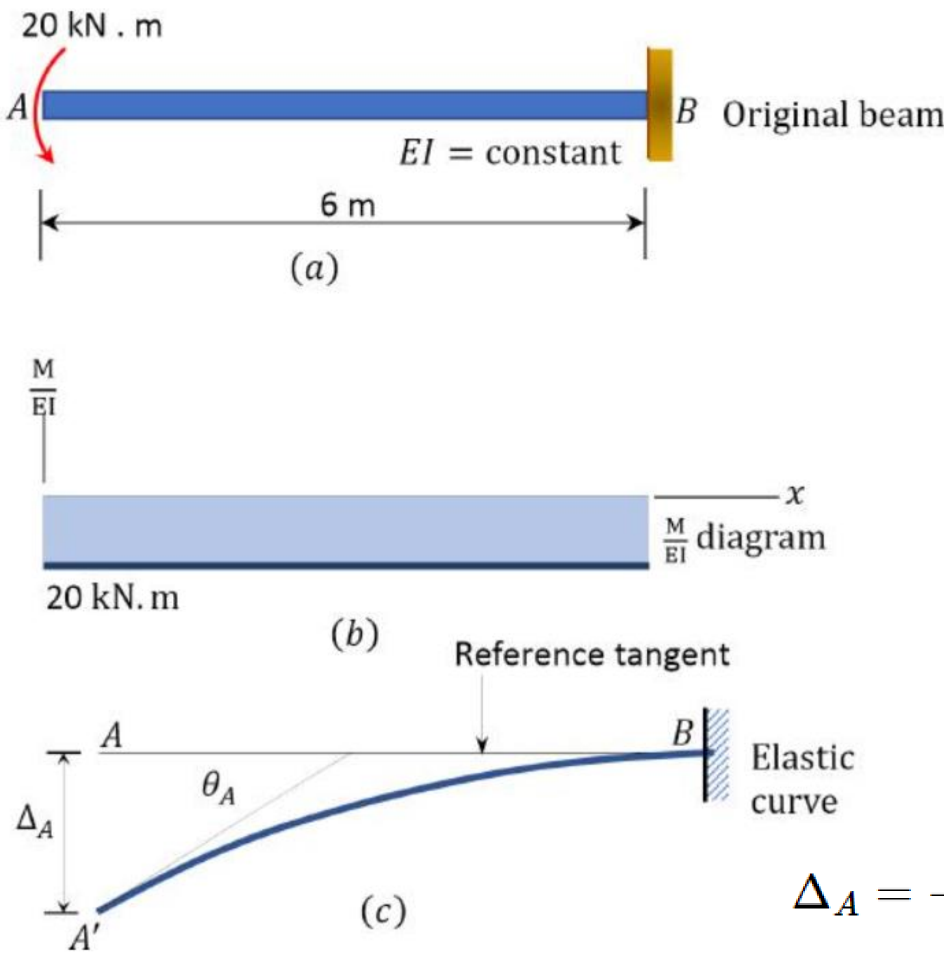
DETERMINE THE SLOPE AT POINTS B AND C OF THE BEAM SHOWN BELOW. TAKE E = 200 GPa AND I = 360 x 10^6 mm^4

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{100\,kNm}{EI}\right)(10\,m) = -\frac{500\,kNm^2}{EI}$$

Substituti ng numerical values of EI, we have :

$$\frac{-500\,kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00694\,rad$$

A cantilever beam shown in Figure is subjected to a concentrated moment at its free end. Using the moment-area method, determine the slope at the free end of the beam and the deflection at the free end of the beam. EI = constant.



$$\theta_A = -\left(\frac{1}{EI}\right)(6)(20) = -\frac{120}{EI}$$

$$\Delta_A = -\left(\frac{1}{EI}\right)(6)(20)(3) = -\frac{360}{EI} \qquad \Delta_A = \frac{360}{EI} \downarrow$$

- A cantilever of length 2 m carries a point load of 20 kN at the free end and another load of 20 kN at the centre. If $E = 10^5\ N/mm^2$ and $I = 10^8\ mm^4$ then determine, the slope and deflection of the cantilever at the free end.

Conjugate beam method

- The slopes and deflections may be obtained from various methods discussed earlier like double integration method, Macaulay's method, moment area method etc.
- But these methods become laborious, when applied to beams whose flexural rigidity (EI) is not uniform throughout the length of the beam
- The slopes and deflections of such beams may be easily obtained by conjugate beam method
- **Conjugate beam is an imaginary beam for which the load diagram is the $\frac{M}{EI}$ diagram**

Conjugate beam method

Conjugate Beam Theorem I :

"*The slope at any section of a loaded beam relative to the original axis of the beam, is equal to the shear in the conjugate beam at the corresponding section.*"

We know that, $\text{load} = w = \frac{M}{EI}$

$$\therefore \quad \text{Shear} = S_x = \int_0^x w \cdot dx = \int_0^x \frac{M}{EI}\, dx$$

But, $$\int_0^x \frac{M}{EI}\, dx = \int_0^x \frac{d^2 y}{dx^2} = \frac{dy}{dx} = slope$$

Conjugate Beam Theorem II :

"*The deflection at any given section of a loaded beam, relative to the original position, is equal to the bending moment at the corresponding section of the conjugate beam.*"

We know that, shear $S_x = \int_0^x \frac{M}{EI}\, dx$

$$\therefore \text{ Bending moment, } M_x = \int_0^x S_x \cdot dx = \int_0^x \int_0^x \frac{M}{EI}\, dx$$

But, $$\int_0^x \int_0^x \frac{M}{EI}\, dx = \int_0^x \int_0^x \frac{d^2 y}{dx^2} = \int_0^x \frac{dy}{dx} = y = deflection$$ *...Proved*

The following points are worth noting for the **conjugate beam method:**

(*i*) This method can be *directly used only for simply supported beams*.

(*ii*) In this method for cantilevers and fixed beams, artificial constraints need to be applied to the conjugate beam so that it is supported in a manner consistent with the constraints of the real beam.

Conjugate beam method

We have, $EI \cdot \frac{d^2y}{dx^2} = M$ or $\frac{d^2y}{dx^2} = \frac{M}{EI}$

Differentiating it, $EI \cdot \frac{d^3y}{dx^3} = \frac{dM}{dx} = F$

Differentiating it again, $EI \cdot \frac{d^4y}{dx^4} = \frac{dF}{dx} = -w$

$$\frac{d^4y}{dx^4} = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right) = -\frac{w}{EI}$$

$$\frac{d^2}{dx^2}\left(\frac{M}{EI}\right) = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2M}{dx^2} = -w$$

Conjugate beam method

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad \text{———————} \qquad \text{(i)}$$

$$\frac{d^2M}{dx^2} = -w \qquad \text{———————} \qquad \text{(ii)}$$

- Thus as indicated by (ii), if *w* indicates the actual loading, and a bending moment diagram is drawn, it provides the bending moment at any cross-section of the beam.

- In a similar way it may be said from (i) that if the bending moment diagram (*M/EI*) is assumed as the loading diagram on the beam (the beam is known as *conjugate beam*) and a new bending moment diagram is constructed from this, the diagram will be a *deflection curve.*

Conjugate beam method

A similar analogy for the slope can also be deduced

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI}$$

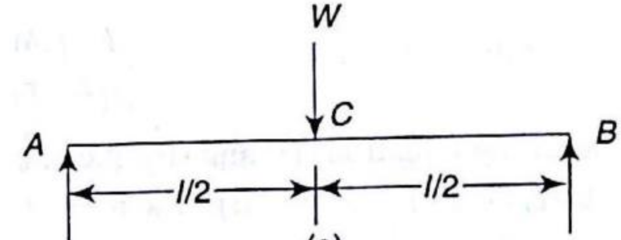
or $$\frac{d}{dx}(slope) = \frac{M}{EI} \quad \text{———} \quad \text{(iii)}$$

Also, $$\frac{dF}{dx} = -w \quad \text{———} \quad \text{(iv)}$$

Thus shear force diagram drawn with *M/EI* as loading will provide the slope at any section.

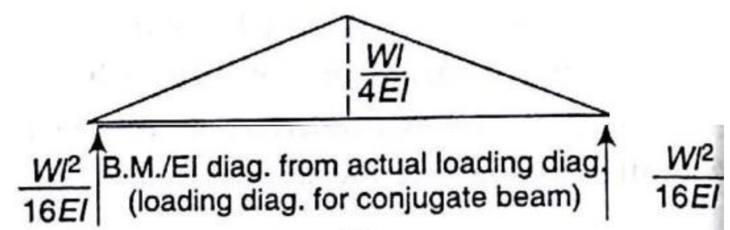
Conjugate beam method – Problem1

Find expressions for the central deflection and the slope at the ends of a simply supported beam carrying a central load by conjugate beam method.



(a)

maximum bending moment at the centre is $Wl/4$,



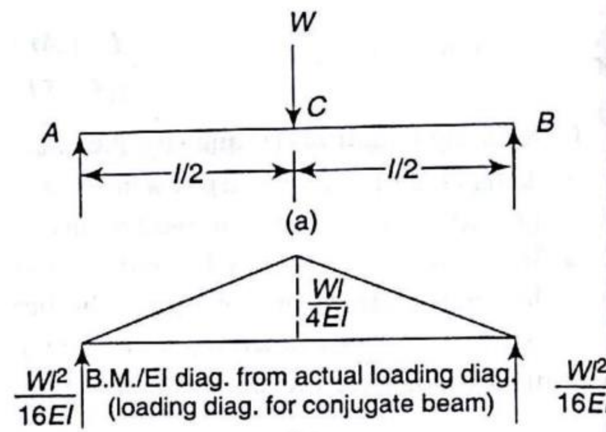
(b)

Now, in the conjugate beam method, this diagram is to be considered as loading diagram

Conjugate beam method – Problem1

first we need to find the reaction on the supports.

$$R_a = R_b = \frac{Wl}{4EI} \times \frac{l}{2} \times \frac{1}{2} = \frac{Wl^2}{16EI}$$



Deflections

Deflection y at any point at a distance x from A

= bending moment due to load on the conjugate beam

$$= \frac{Wl^2}{16EI}x - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{Wl^2}{16EI}x - \frac{W}{12EI}x^3 = \frac{W}{48EI}(3l^2x - 4x^3)$$

Maximum deflection at the centre $= \frac{W}{48EI}\left[3l^2 \cdot \frac{l}{2} - 4\left(\frac{l}{2}\right)^3\right] = \frac{Wl^3}{48EI}$

Slopes

Slope at any point at a distance x from A

= Shearing force at the point due to load on the conjugate beam

$$= \frac{Wl^2}{16EI} - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2}$$

Slope at the ends $= \frac{Wl^2}{16EI}$ .....$(x = 0)$

Problem 14.1. *A simply supported beam of length 5 m carries a point load of 5 kN at a distance of 3 m from the left end. If $E=2\times10^5$ N/mm^2 and $I=10^8$ mm^4, determine the slope at the left support and deflection under the point load using conjugate beam method.*

Sol. Given :

Length, $L=5$ m

Point load, $W=5$ kN

Distance AC, $a=3$ m

Distance BC, $b=5-3=2$ m

Value of $E=2\times10^5 \text{ N/mm}^2=2\times10^5\times10^6 \text{ N/m}^2$

$=2\times10^5\times10^3 \text{ kN/m}^2$

$=2\times10^8 \text{ kN/m}^2$

Value of $I=1\times10^8 \text{ mm}^4=10^{-4} \text{ m}^4$

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking moments about A, we get

$$R_B\times5=5\times3$$

$$\therefore \quad R_B=\frac{5\times3}{5}=3 \text{ kN}$$

and R_A = Total load $-R_B$

$=5-3=2$ kN

Now B.M. diagram is ...

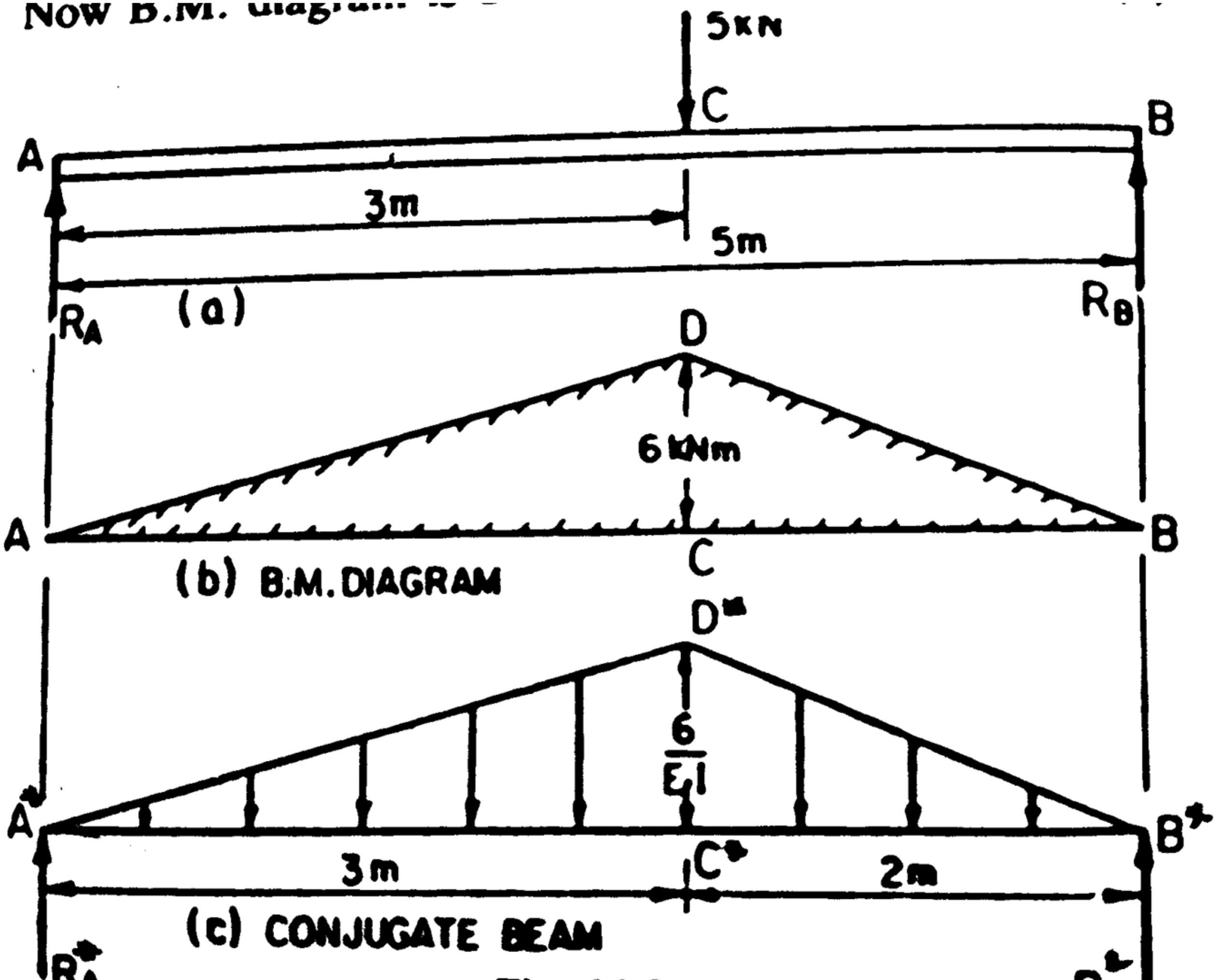


Fig. 14.3

The B.M. at $A = 0$

B.M. at $B = 0$

B.M. at $C = R_A \times 3 = 2 \times 3 = 6$ kNm.

Now B.M. diagram is drawn as shown in Fig. 14.3 (*b*).

Now construct the conjugate beam as shown in Fig. 14.3 (*c*). The vertical load at C^* on conjugate beam

$$= \frac{\text{B.M. at } C}{EI} = \frac{6 \text{ kNm}}{EI}$$

Now calculate the reaction at A^* and B^* for conjugate beam

Let R_A^* = Reaction at A^* for conjugate beam

R_B^* = Reaction at B^* for conjugate beam.

Taking moments about A^*, we get

$R_B^* \times 5$ = Load on $A^*C^*D^*$ × distance of C.G. of $A^*C^*D^*$ from A^*

+ Load on $B^*C^*D^*$ × Distance of C.G. of $B^*C^*D^*$ from A^*

$$= \left(\frac{1}{2} \times 3 \times \frac{6}{EI}\right) \times \left(\frac{2}{3} \times 3\right) + \left(\frac{1}{2} \times 2 \times \frac{6}{EI}\right) \times \left(3 + \frac{1}{3} \times 2\right)$$

$$= \frac{18}{EI} + \frac{6}{EI} \times \frac{11}{3} = \frac{8}{EI} + \frac{22}{EI} = \frac{40}{EI}$$

$$\therefore \quad R_B^* = \frac{40}{EI} \times \frac{1}{5} = \frac{8}{EI}$$

$\therefore$ $R_A^* = \text{Total load } (i.e., \text{ load } A^*B^*D^*) - R_B^*$

$$= \left(\frac{1}{2} \times 5 \times \frac{6}{EI}\right) - \frac{8}{EI}$$

$$= \frac{15}{EI} - \frac{8}{EI} = \frac{7}{EI}$$

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at A

y_C = Deflection at C for the given beam

Then according to conjugate beam method,

θ_A = Shear force at A^* for conjugate beam = R_A^*

$$= \frac{7}{EI} = \frac{7}{2 \times 10^8 \times 10^{-4}} \quad (\because E = 2 \times 10^8 \text{ kN/m}^2 \text{ and } I = 10^{-4} \text{ m}^4)$$

= **0.00035 radians. Ans.**

y_C = B.M. at C^* for conjugate beam

$= R_A^* \times 3$ – Load $A^*C^*D^*$ × Distance of C.G. of $A^*C^*D^*$ from C^*

$$= \frac{7}{EI} \times 3 - \left(\frac{1}{2} \times 3 \times \frac{6}{EI}\right) \times \left(\frac{1}{3} \times 3\right)$$

$$= \frac{21}{EI} - \frac{9}{EI} - \frac{12}{EI}$$

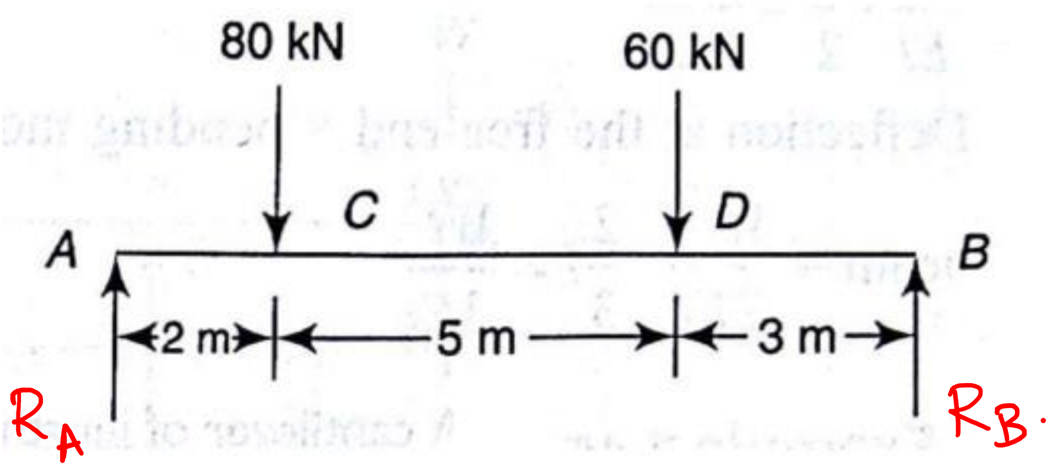
$$= \frac{12}{2 \times 10^8 \times 10^{-4}} = \frac{6}{10^4} \text{ m} = \frac{6 \times 1000}{10000} \text{ mm} = \mathbf{0.6 \text{ mm}}. \quad \textbf{Ans.}$$

Conjugate beam method – Problem2

R_A R_B

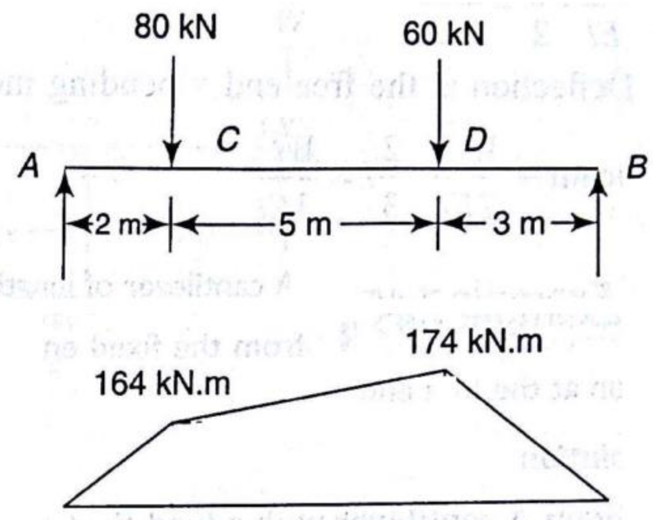
$\Sigma F = 0$

$\Sigma M_A = 0$



Conjugate beam method – Problem2

A 10 m long simply supported beam *AB* carries loads of 80 kN and 60 kN at 2 m and 7 m respectively from *A*. $E = 200$ GPa and $I = 150 \times 10^6$ mm^4. Determine the deflection and slope under the loads using conjugate beam method.



Taking moments about A,

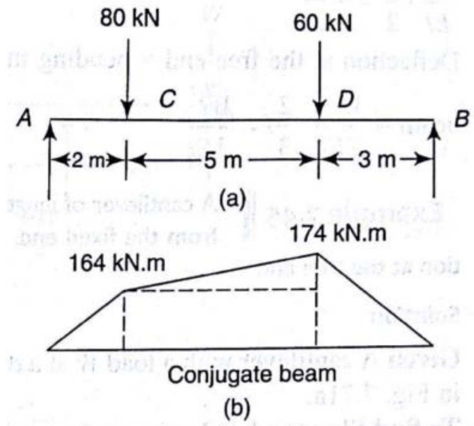
$$10\,R_b = 80 \times 2 + 60 \times 7$$

or $$R_b = 58 \text{ kN}$$

$$R_a = 80 + 60 - 58 = 82 \text{ kN}$$

Bending moment at $C = 82 \times 2 = 164$ kN·m
Bending moment at $D = 58 \times 3 = 174$ kN·m

Conjugate beam method – Problem2



Conjugate beam

Bending moment (conjugate beam) diagram is shown in Fig. 7.69b.

Taking moments about *B* to find the reaction at *A* from conjugate loads,

$$10\,R_a = \left(164 \times 2 \times \frac{1}{2}\right)\left(\frac{2}{3} + 8\right) + 164 \times 5\left(3 + \frac{5}{2}\right) + (174 - 164) \times 5 \times \frac{1}{2}\left(3 + \frac{5}{3}\right) + 174 \times 3 \times \frac{1}{2} \times 2$$

$$10\,R_a = 1421.3 + 4510 + 116.7 + 522 \quad \text{or} \quad R_a = 657$$

$$R_b = 164 \times (2/2) + 164 \times 5 + (174 - 164) \times (5/2) + 174 \times (3/2) - 657 = 613$$

For conjugate beam

Shearing force at *C* = 657 – 164 × (2/2) = 493

Shearing force at *D* = –613 + 174 × (3/2) = –352

Bending moment at *C* = 657 × 2 – 164 × (2/3) = 1204.7

Bending moment at *D* = 613 × 3 – 174 × (3/2) × 1 = 1578

Conjugate beam method – Problem2

Slope and deflection

$EI = 200 \times 10^6 \times (150 \times 10^{-6}) = 30\,000$ kN·m^2

Slope at C = 493/30 000 = 0.0164 rad

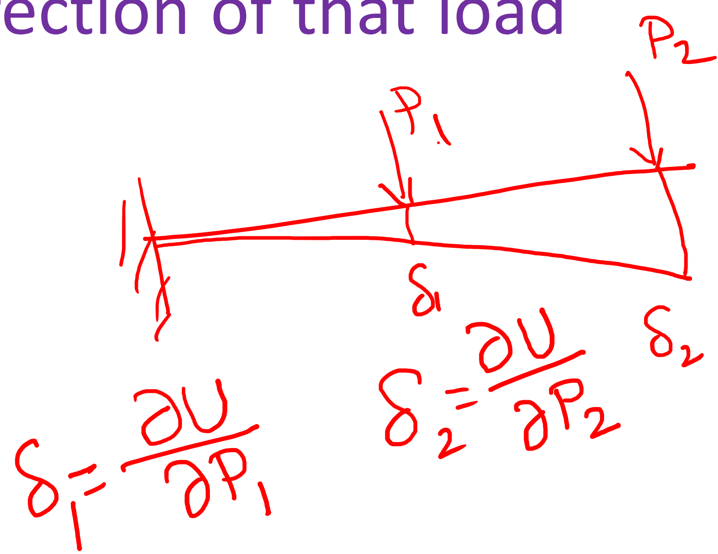
Slope at D = 352/30 000 = 0.0117 rad

Deflection at C = 1204.7/30 000 = 0.04016 m = 40.16 mm

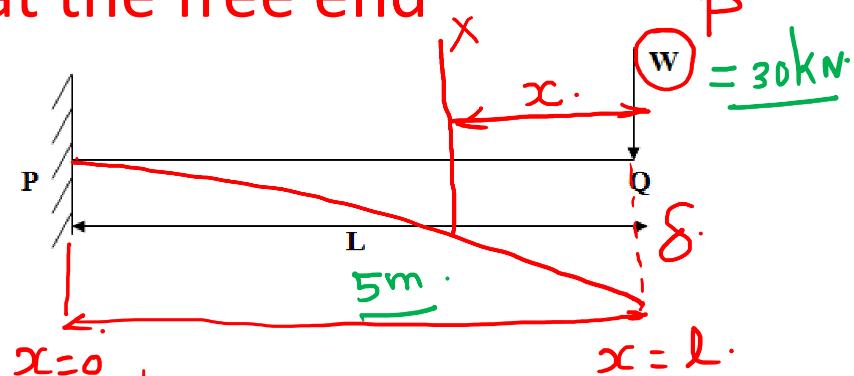
Deflection at D = 1578/30 000 = 0.0526 m = 52.26 mm

Strain energy in bending
Castigliano's first theorem

- Castigliano's theorem provides a means for finding the deflections of a structure from the strain energy of the structure
- If a structure is subjected to a number of point loads (or couples) the partial derivative of the total strain energy with respect to any load (or couple) provides the deflection in the direction of that load (or couple)
- Mathematically $\delta_i = \frac{\partial U}{\partial P_i}$
- Strain energy $U_b = \int \frac{M^2 dx}{2EI}$



- Find the deflection at the free end of a cantilever which carries a point load at the free end



Strain Energy Method

$$\delta = \frac{\partial U}{\partial P} = \frac{\partial U}{\partial W}$$

$$U = \int_0^l \frac{M^2\,dx}{2EI}$$

$$M = Wx$$

$$U = \int_0^l \frac{(Wx)^2\,dx}{2EI}$$

$$= \int_0^l \frac{W^2}{2EI} x^2\,dx$$

$$U = \frac{W^2}{2EI}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{W^2}{2EI}\left(\frac{l^3}{3} - 0\right)$$

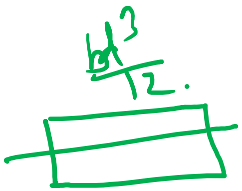
$$= \frac{W^2 l^3}{6EI}$$

$$\delta = \frac{\partial U}{\partial W}$$

$$= \frac{\partial}{\partial W}\left(\frac{W^2 l^3}{6EI}\right)$$

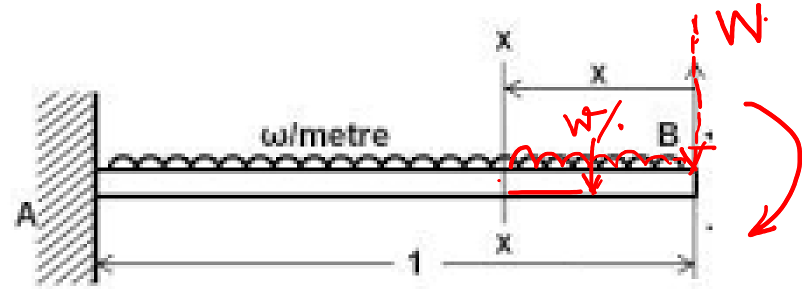
$$= \frac{2Wl^3}{6EI}$$

$$\delta = \frac{Wl^3}{3EI}$$



EI

- Determine the strain energy of a cantilever with uniformly distributed load . Also find the deflection at free end.



$$\delta = \frac{\partial U}{\partial P} \qquad U = \int \frac{M^2\,dx}{2EI}$$

$$M = \left(Wx + \frac{wx^2}{2}\right)$$

$$U = \int_0^l \frac{\left(Wx + \frac{wx^2}{2}\right)^2}{2EI}\,dx$$

$$\delta = \frac{1}{2EI}\int_0^l 2\left(Wx + \frac{wx^2}{2}\right)x\,dx$$

$$= \frac{1}{EI}\int_0^l \left(Wx^2 + \frac{wx^3}{2}\right)dx$$

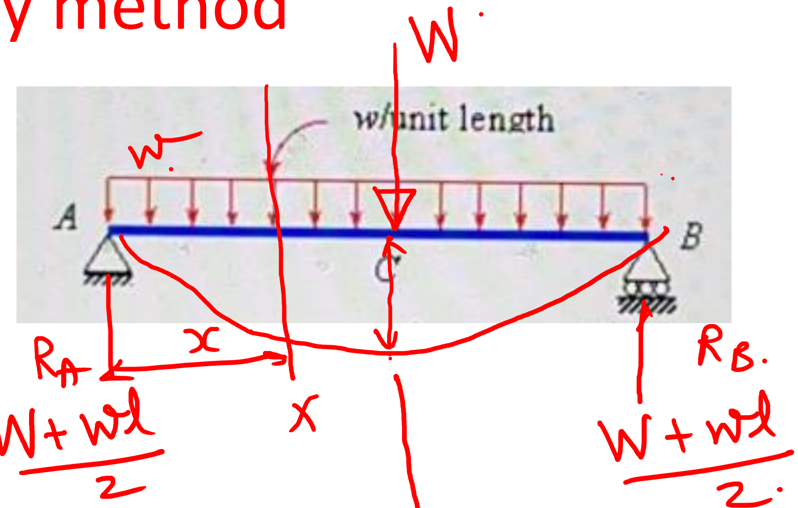
$$\delta = \frac{1}{EI}\left[\frac{Wx^3}{3} + \frac{wx^4}{8}\right]_0^l$$

$$= \frac{1}{EI}\left[\frac{Wl^3}{3} + \frac{wl^4}{8}\right]$$

$$\delta = \frac{\partial U}{\partial W} \qquad = W = 0$$

$$\delta = \frac{wl^4}{8EI}$$

- Determine the maximum deflection of a simply supported beam of span l carrying a load of w per unit length using strain energy method



$$R_A + R_B = W + wl$$

$$M_x = R_A x - w\frac{x^2}{2}$$

$$= \left(\frac{W+wl}{2}\right)x - \frac{wx^2}{2}$$

$$U = \int \frac{M^2\,dx}{2EI} = 2 \times \int_0^{l/2} \left[\left(\frac{W+wl}{2}\right)x - \frac{wx^2}{2}\right]^2 \frac{dx}{2EI}$$

$$= \frac{1}{EI}\int_0^{l/2} \left[\left(\frac{W+wl}{2}\right)x - \frac{wx^2}{2}\right]^2 dx$$

$$U = \frac{1}{EI}\int_0^{l/2}\left[\left(\frac{W+wl}{2}\right)x - \frac{wx^2}{2}\right]^2 dx.$$

$$\delta = \frac{\partial U}{\partial W} = \frac{1}{EI}\int_0^{l/2} 2\left[\left(\frac{W+wl}{2}\right)x - \frac{wx^2}{2}\right] dx.$$

$$= \frac{1}{EI}\left[\left(\frac{Wx^2}{2} + \frac{wlx^2}{2} - \frac{wx^3}{6}\right)\right]_0^{l/2}$$

$$= \frac{1}{2EI}\left[\frac{Wl^3}{24} + \frac{wl^4}{24} - \frac{wl^4}{64}\right]$$

$W = 0$

$$\delta = \frac{5}{384}\frac{wl^4}{EI}$$


THANK
YOU

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STRENGTH OF MATERIALS FOR MECHANICAL ENGINEERS (CE8395)

UNIT-5

Thin cylinders, spheres and thick cylinders

by

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Syllabus

UNIT V THIN CYLINDERS, SPHERES AND THICK CYLINDERS 9

Stresses in thin cylindrical shell due to internal pressure circumferential and longitudinal stresses and deformation in thin and thick cylinders – spherical shells subjected to internal pressure – Deformation in spherical shells – Lame's theorem.

Course objective:

To study the stresses and deformations induced in thin and thick shells

Course outcome:

After completion of this unit students should be able to:

Analyze and design thin and thick shells for the applied internal and external pressures

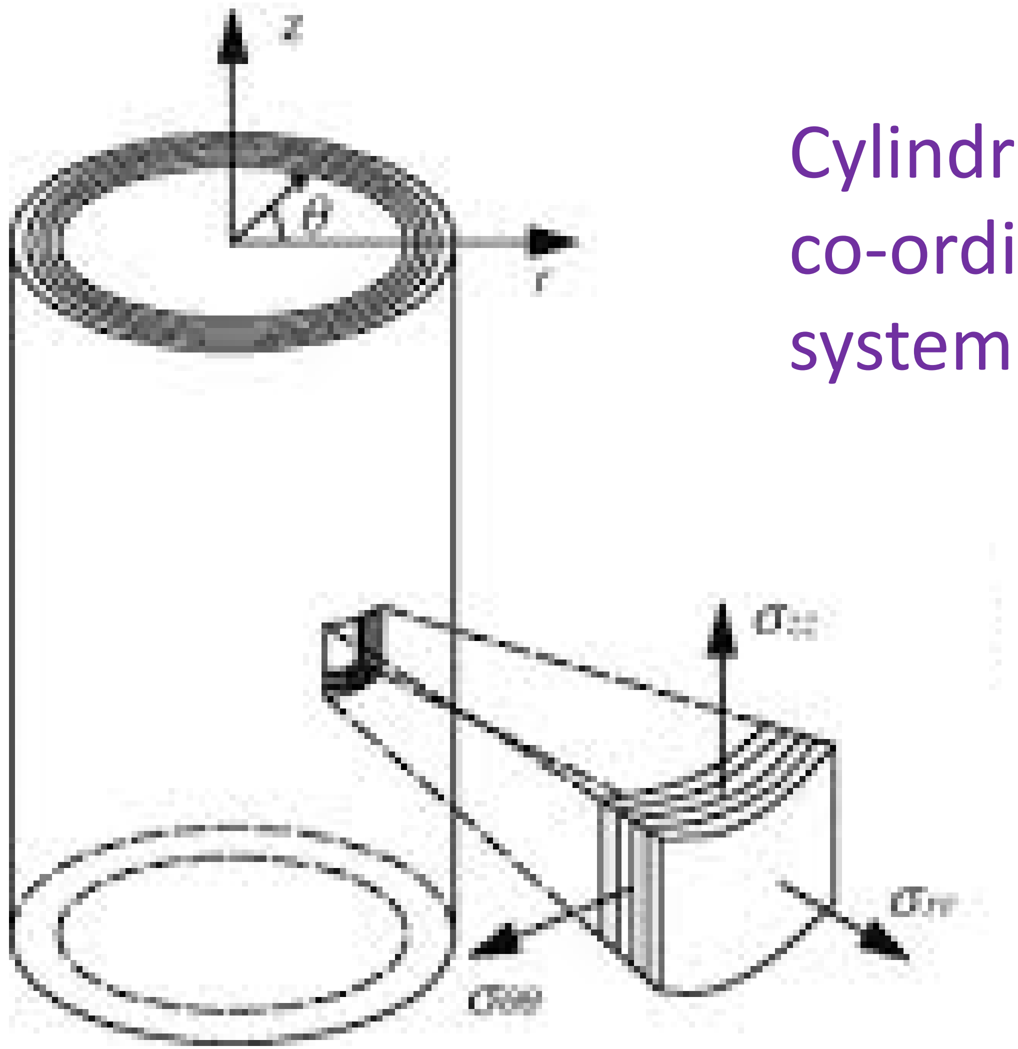
Thin cylinders and spheres

- If the **thickness to internal diameter ratio** is less than **1/20**, then it is **thin cylinder**

Stresses acting on thin cylinder

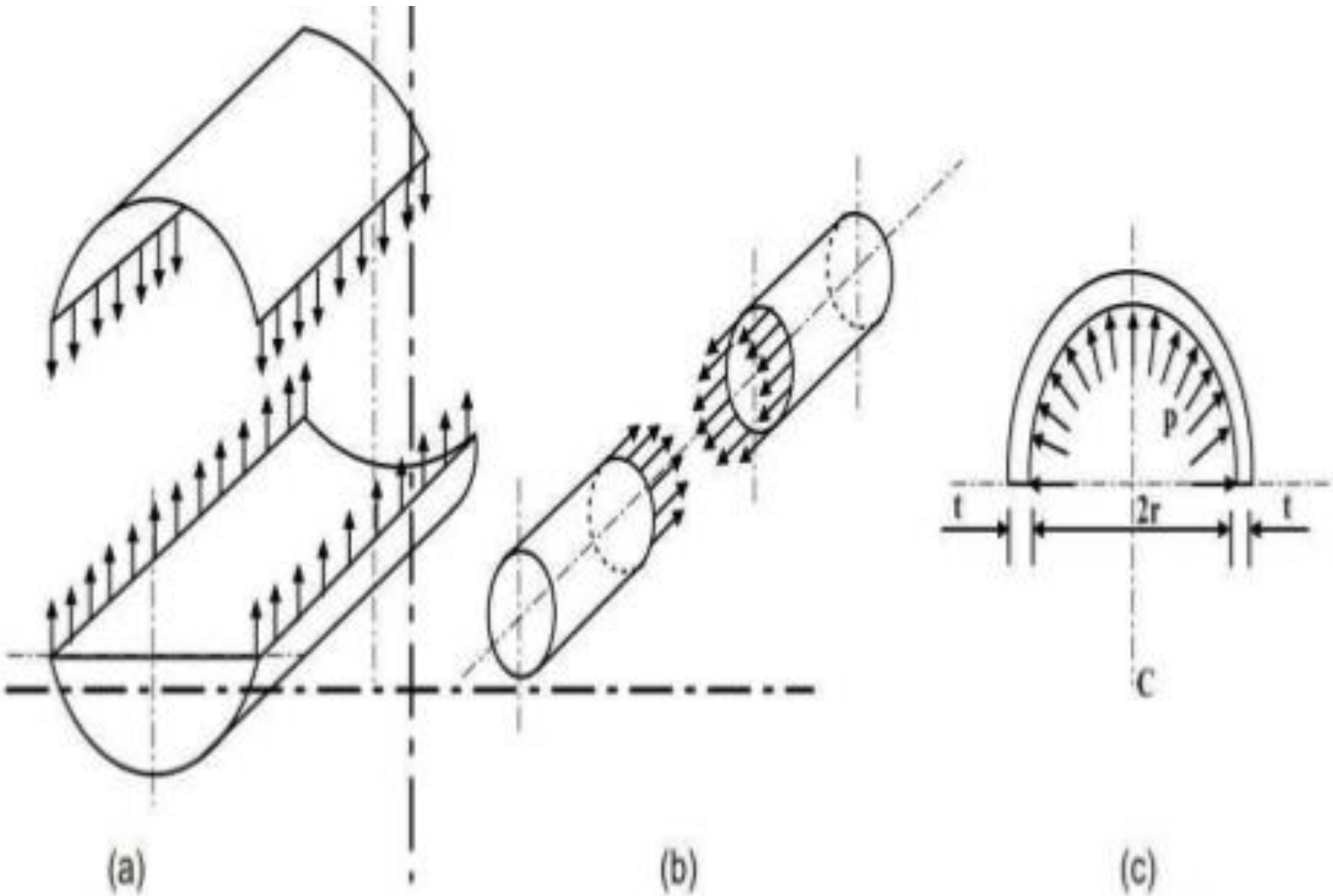
- When a cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up within the walls of the cylinder
  - **Hoop or circumferential stress**- tensile in nature, tends to increase the diameter
  - **Longitudinal or axial stress** – tensile in nature, tends to increase the length
  - **Radial stress (radial pressure)**– compressive in nature, its magnitude is equal to fluid pressure on the inside wall and zero on the outside wall if it is open to atmosphere

Stresses developed in thin cylinders



Cylindrical co-ordinate system

Stresses developed in thin cylinders

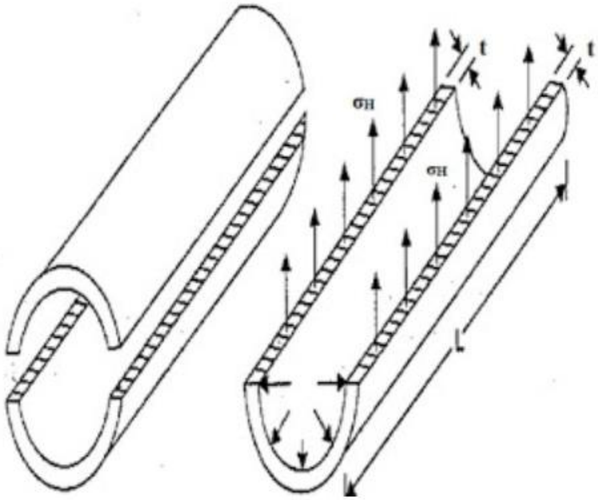


9.1.1.1F- (a) Circumferential stress (b) Longitudinal stress and (c) Radial stress developed in thin cylinders.

Thin cylinder

- In case of thin cylinders subjected to low internal pressures, the radial stress is uniform over the thickness and hence ignored
- So the stresses considered here
  - **Hoop or circumferential**
  - **Longitudinal or axial stress**

Hoop or circumferential stress



$$\sigma_H = \frac{Pd}{2t}$$

σ_H – *cylinder hoop stress in Pa*

P – *internal pressure in Pa*

d – *cylinder inside diameter in m*

t – *wall thickness in m*

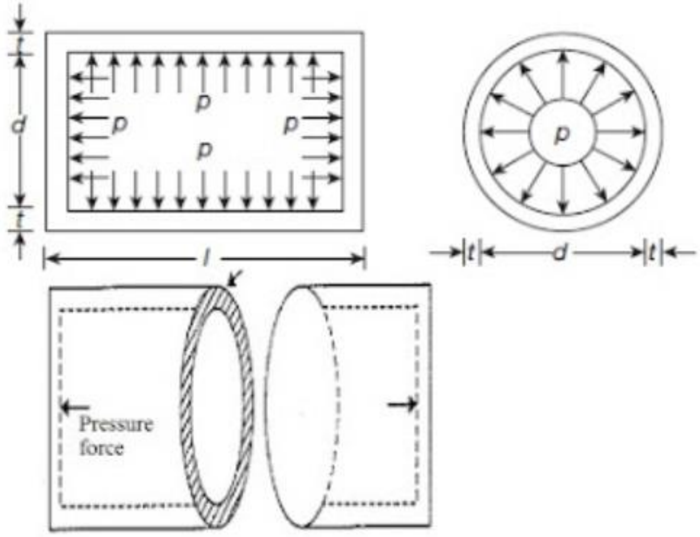
Force due to internal fluid pressure = Resisting force due to circumferential stress

$$P \times d \times L = \sigma_H \times 2\,L\,t$$

$$\sigma_H = P \times d / (2\,t)$$

$$\sigma_H = \frac{Pd}{2t}$$

Longitudinal or axial stress



Force due to internal fluid pressure = Resisting force due to longitudinal stress

$$P \times (\pi/4)\, d^2 = \sigma_L \times \pi\, d\, t$$

$$\sigma_L = P \times d / (4\, t)$$

$$\sigma_L = \frac{Pd}{4t}$$

Stresses in cylindrical shell

- **Circumferential or hoop stress**

$$\sigma_c = \frac{\mathbf{pd}}{\mathbf{2t}}$$

- **Longitudinal or axial stress**

$$\sigma_l = \frac{\mathbf{pd}}{\mathbf{4t}}$$

(Longitudinal stress = half of circumferential stress)

- **Maximum shear stress**

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\frac{p\,\mathrm{d}}{2t} - \frac{p\,\mathrm{d}}{4t}}{2} = \frac{pd}{8t}$$

Efficiency of the joint

- In the longitudinal joint circumferential stress is developed, whereas in circumferential joint longitudinal stress is developed
- $\sigma_c = \frac{\text{pd}}{2\text{t}\eta_l}$

  Where η_l - efficiency of longitudinal joint
- $\sigma_l = \frac{\text{pd}}{4\text{t}\eta_c}$

Where η_c - efficiency of circumferential joint

Strains in cylindrical shell

- **Circumferential strain**

$$\epsilon_c = \frac{\sigma_c}{E} - \mu\frac{\sigma_l}{E}$$

$$\boldsymbol{\epsilon_c = \frac{pd}{2tE}\left(1 - \frac{\mu}{2}\right)}$$

- **Longitudinal strain**

$$\epsilon_l = \frac{\sigma_l}{E} - \mu\frac{\sigma_c}{E}$$

$$\boldsymbol{\epsilon_l = \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right)}$$

- **Volumetric strain**

$$\varepsilon_v = \frac{\delta l}{l} + 2\frac{\delta d}{d} = \varepsilon_l + 2\varepsilon_c$$

$$\boldsymbol{\varepsilon_v = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right)}$$

Effect of internal pressure on the dimensions of a thin cylindrical shell

- Change in diameter

$$\delta d = \frac{pd}{2tE}\left(1 - \frac{\mu}{2}\right) \times d = \frac{pd^2}{2tE}\left(1 - \frac{\mu}{2}\right)$$

- Change in length

$$\delta l = \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right) \times l = \frac{pdl}{2tE}\left(\frac{1}{2} - \mu\right)$$

- Change in volume

$$\delta V = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right) \times V = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right) \times \frac{\pi}{4}\mathrm{d}^2 l$$

Problem-1

- A cylindrical shell 3 meters long has 1 metre internal diameter and 15 mm metal thickness. Calculate the circumferential and longitudinal stresses induced and also changes in the dimensions of the shell, if it is subjected to an internal pressure of 150 N/cm^2. Take E = 200x10^5 N/cm^2 and Poisson's ratio = 0.3.

$l = 3\,m = 3000\,mm.$

$d = 1\,m = 1000\,mm. = 100\,cm$

$t = 15\,mm = 1.5\,cm$

$E = 200 \times 10^5\ N/cm^2$

$P = 150\ N/cm^2$

$\nu = \frac{1}{m} = 0.3.$

Circumferential stress

$$\sigma_c = \frac{pd}{2t} = \frac{150 \times 100}{2 \times 1.5}$$

$$= 5000\ N/cm^2$$

Longitudinal stress

$$\sigma_l = \frac{pd}{4t} = \frac{150 \times 100}{4 \times 1.5}$$

$$= 2500\ N/cm^2$$

Then also find the following

Change in diameter

Change in length

Change in volume

Problem-2

- A cylindrical shell 800 mm inner diameter, 3 m long is having 10 mm metal thickness. If the shell is subjected to an internal pressure of 2.5 N/mm^2 , find (i) the change in diameter (ii) the change in length and (iii) the change in volume. Assume the modulus of elasticity and Poisson's ratio of the material of the shell as 200 kN/mm^2 and 0.25 respectively.

$d = 800\,mm$, $l = 3\,m$
$t = 10\,mm$. $= 3000\,mm$.
~~$D = d + 2t = 800 + 2\times10 = 820\,mm$~~
$p = 2.5\ N/mm^2$
$E = 200\ kN/mm^2 = 200\times10^3\ N/mm^2$
$\nu = \frac{1}{m} = 0.25$

i) change in diameter

$$\varepsilon_c = \frac{\delta d}{d} = \frac{pd}{2tE}\left(1 - \frac{1}{2m}\right)$$

$$\frac{\delta d}{800} = \frac{2.5\times800}{2\times10\times200\times10^3}\left(1 - \frac{1}{2}\times0.25\right)$$

$$\delta d = 0.35\ mm$$

ii) change in length

$$\varepsilon_l = \frac{\delta l}{l} = \frac{pd}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right)$$

$$\frac{\delta l}{3000} = \frac{2.5\times800}{2\times10\times200\times10^3}\left(\frac{1}{2} - 0.25\right)$$

$$\delta l = 0.375\ mm.$$

iii) change in volume

$$\varepsilon_v = \frac{\delta V}{V} = \frac{pd}{2tE}\left(\frac{5}{2} - \frac{2}{m}\right)$$

$$\frac{\delta V}{V} = \frac{2.5\times800}{2\times10\times200\times10^3}\left(\frac{5}{2} - 2\times0.25\right)$$

$$= 5\times10^{-4}\,(2)$$

$$V = \frac{\pi}{4}d^2 l = \frac{\pi}{4}(800)^2\times3000$$

$$= 1507964474$$

$$= 1.507\times10^9\ mm^3$$

$$\delta V = 1.508\times10^6\ mm^3$$

Problem-3

A thin cylinder 60mm internal diameter, 225mm long with walls 2.7mm thick is subjected to an internal pressure of $6MN/m^2$. You may assume that $E = 200GN/m^2$ and $v = 0.3$. Calculate:

i. The hoop stress
ii. The longitudinal stress
iii. The change in length
iv. The change in diameter

Hoop stress:

$$\sigma_\theta = \frac{p_i d}{2t} = \frac{(6 \times 10^6)(60 \times 10^{-3})}{2 \times 2.7 \times 10^{-3}} = 66.7MN/m^2$$

Longitudinal stress:

$$\sigma_L = \frac{p_i d}{4t} = \frac{(6 \times 10^6)(60 \times 10^{-3})}{4 \times (2.7 \times 10^{-3})} = 33.3MN/m^2$$

Change in length:

$$\delta L = \frac{p_i d}{4tE}(1 - 2v)L = \frac{(6 \times 10^6)(60 \times 10^{-3})}{4 \times (2.7 \times 10^{-3}) \times (200 \times 10^9)}(1 - 0.6)(225 \times 10^{-3})$$

$$\delta L = 15 \times 10^{-6} m$$

Change in diameter:

$$\delta d = \frac{p_i d^2}{4tE}(2 - v) = \frac{(6 \times 10^6)(60 \times 10^{-3})^2}{4 \times (2.7 \times 10^{-3}) \times (200 \times 10^9)}(2 - 0.3)$$

$$\delta d = 17 \times 10^{-6} m$$

Problem-4

A 1m long thin cylinder has an internal diameter of 200mm with a wall thickness of 3mm. If it found to undergo a change to its internal volume of $9 \times 10^{-6} m^3$ when subject to an internal pressure p. You may assume that $E = 210GN/m^2$ and $v = 0.3$. Calculate the hoop and longitudinal stresses.

We have:

$$\delta V = \frac{p_i d}{4tE}(5 - 4v)V$$

Original volume, V:

$$V = \frac{\pi}{4}(200 \times 10^{-3})^2 \times 1 = 31.4 \times 10^{-3} m$$

$$p_i = \frac{\delta V 4tE}{d(5 - 4v)V} = \frac{(9 \times 10^{-6}) \times 4 \times (3 \times 10^{-3}) \times (210 \times 10^{9})}{(200 \times 10^{-3})(5 - 1.2)(31.4 \times 10^{-3})}$$

$$p_i = 0.95MN/m^2$$

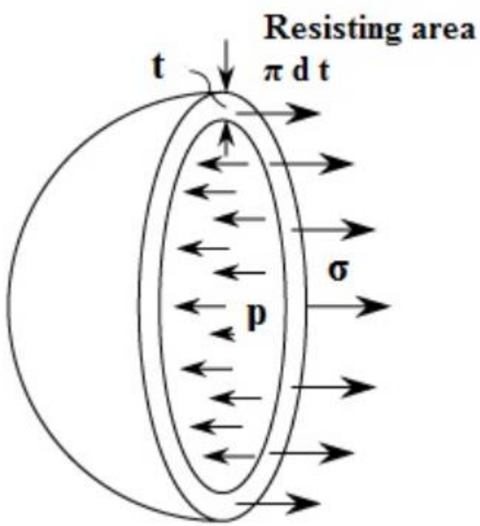
Hoop stress:

$$\sigma_\theta = \frac{p_i d}{2t} = \frac{(0.95 \times 10^{6})(200 \times 10^{-3})}{2(3 \times 10^{-3})} = 31.66MN/m^2$$

Longitudinal stress:

$$\sigma_L = \frac{p_i d}{4t} = \frac{(0.95 \times 10^{6})(200 \times 10^{-3})}{4(3 \times 10^{-3})} = 15.83MN/m^2$$

Thin spherical shell



P = Internal fluid pressure

d = Internal diameter of thin spherical shell

t = Thickness of the wall of thin spherical shell

σ = Circumferential stress or hoop stress developed in the wall

Force due to internal fluid pressure = Resisting force due to longitudinal stress

$$P \times (\pi/4)\, d^2 = \sigma \times \pi\, d\, t$$

$$\boldsymbol{\sigma = P \times d / (4\, t)}$$

Thin spherical shell

- Hoop or circumferential stress
- $\sigma_c = \frac{\text{pd}}{4\text{t}}$
- Strain in any direction
- $\varepsilon = \frac{\sigma_c}{E} - \mu \frac{\sigma_c}{E}$
- $\varepsilon = \frac{\delta d}{d} = \frac{\text{pd}}{4\text{tE}}(1 - \mu)$
- Change in diameter
- $\delta d = \frac{\text{pd}^2}{4\text{tE}}(1 - \mu)$
- Volumetric strain
- $\varepsilon_v = \frac{\delta V}{V} = \frac{3\text{pd}}{4\text{tE}}(1 - \mu)$

Problem (Thin spherical shell)

- A spherical shell of 2 m diameter is made up of 10 mm thick plates. Calculate the change in diameter and volume of the shell, when it is subjected to an internal pressure of 1.6 MPa. Take E = 200 GPa and 1/m = 0.3.

$d = 2m = 2000 \text{ mm}$

$t = 10 \text{ mm}$

$p = 1.6 \text{ MPa} = 1.6 \times 10^6 \text{ N/m}^2$

$= 1.6 \times 10^6 \text{ N/}(10^3)^2 \text{mm}^2$

$= 1.6 \text{ N/mm}^2$

$E = 200 \text{ GPa} = 200 \times 10^9 \text{ N/m}^2$

$= 200 \times 10^3 \text{ N/mm}^2$

$\mu = \frac{1}{m} = 0.3$

change in diameter

$\varepsilon = \frac{\delta d}{d} = \frac{pd}{4tE}\left(1 - \frac{1}{m}\right)$

$\frac{\delta d}{2000} = \frac{1.6 \times 2000}{4 \times 10 \times 200 \times 10^3}(1 - 0.3)$

$\delta d = 0.56 \text{ mm}$

change in volume

$\varepsilon_v = \frac{\delta V}{V} = \frac{3pd}{4tE}\left(1 - \frac{1}{m}\right)$

$\delta V = 3.518 \times 10^6 \text{ mm}^3$

$V = \frac{\pi}{6} d^3 = \frac{\pi}{6}(2000)^3$

$= 4.1887 \times 10^9 \text{ mm}^3$

Thick cylinders and spheres

- Earlier we have analysed thin cylinders, which are basically used for low internal pressure
- The internal pressure was considered to be negligible in comparison to the circumferential stress and longitudinal stress
- Thick cylinders are designed to carry high internal pressure
- Hence radial compressive stress can not be neglected
- In addition, circumferential stress which was assumed to be constant, is no longer constant, but varies along the thickness

Thick cylinders and spheres

- The problem of thick cylinders is some what complex in nature
- It was first solved by French scientist Gabriel Lame in 1833
- His analysis is commonly known as Lame's theory

Thick cylinders – Lame's theory

- Radial pressure

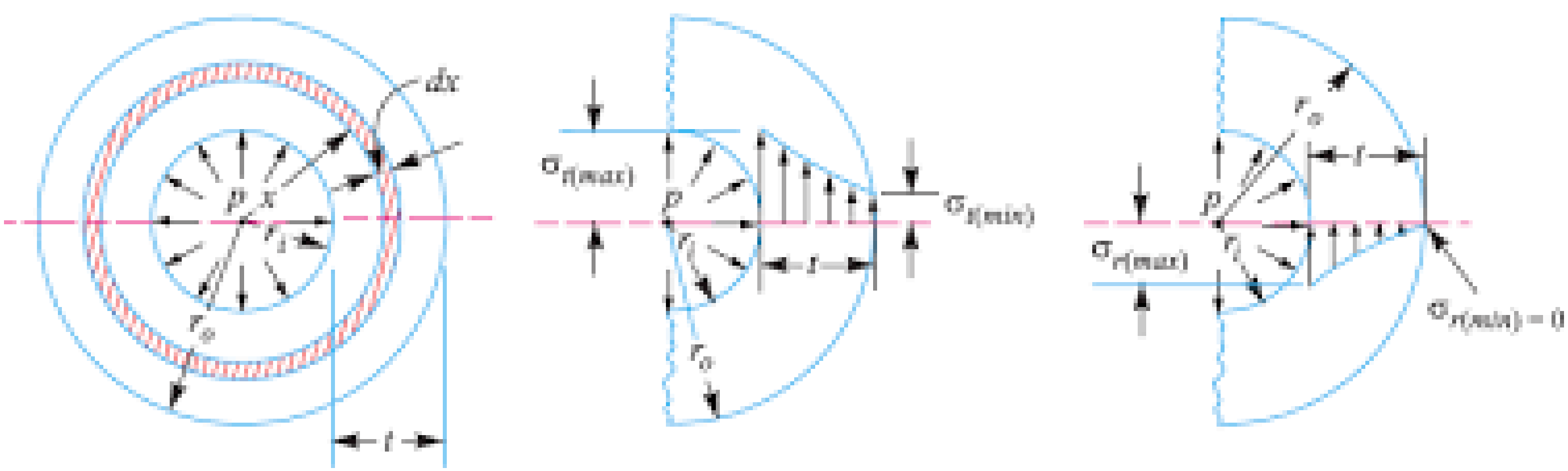
$$p_x = \frac{b}{x^2} - a$$

- Hoop stress

$$\sigma_x = \frac{b}{x^2} + a$$

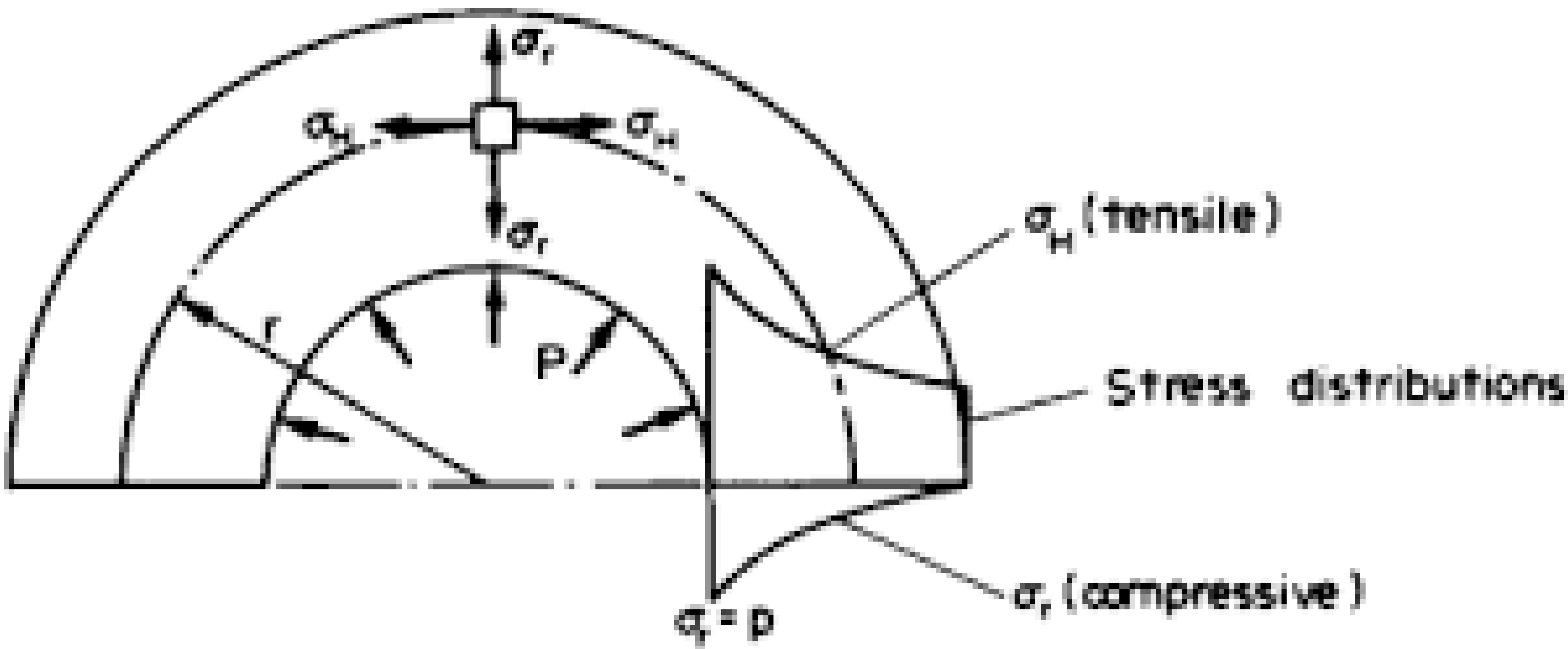
Lame's equations

- Where a and b are constants



(a) Thick cylindrical shell. (b) Tangential stress distribution. (c) Radial stress distribution.

Thick cylinders – Lame's theory



$\sigma_H = A + B/r^2$ $\sigma_r = A - B/r^2$

Thick sphere – Lame's theory

- Radial pressure

$$p_x = \frac{2b}{x^3} - a$$

- Hoop stress

$$\sigma_x = \frac{b}{x^3} + a$$

(Lame's equations)

Where a and b are constants

Thick cylinder – Problem-1

- A cast iron pipe has 20 mm internal radius and 50 mm metal thickness, and carries water under pressure of 5 N/mm^2. Calculate the maximum and minimum intensities of circumferential stress and sketch the distribution of circumferential stress and radial pressure across the section.

Formula:

Radial pressure

(max. at inner radius and zero at outer radius)

$$p_x = \frac{b}{x^2} - a$$

Hoop stress

(max. at inner radius and min. at outer radius)

$$\sigma_x = \frac{b}{x^2} + a$$

Thick cylinder – Problem-1 contd..

5) $r_2 = 20$ mm

$t = 50$ mm

$r_1 = r_2 + t = 70$ mm

$p = 5\ N/mm^2$

Radial pressure

$$p_x = \frac{b}{x^2} - a$$

at $x = r_1$, $p_x = 0$

$$0 = \frac{b}{70^2} - a$$

$$0 = \frac{b}{4900} - a \quad \text{——— (1)}$$

at $x = r_2$, $p_x = 5\ N/mm^2$

$$5 = \frac{b}{20^2} - a$$

$$5 = \frac{b}{400} - a \quad \text{——— (2)}$$

Solving (1) and (2)

$$(2)-(1) \Rightarrow 5 = \frac{b}{400} - \frac{b}{4900}$$

$$5 = \frac{b(4900 - 400)}{400 \times 4900}$$

$b = 2177.78$

$a = 0.4444$

Circumferential stress

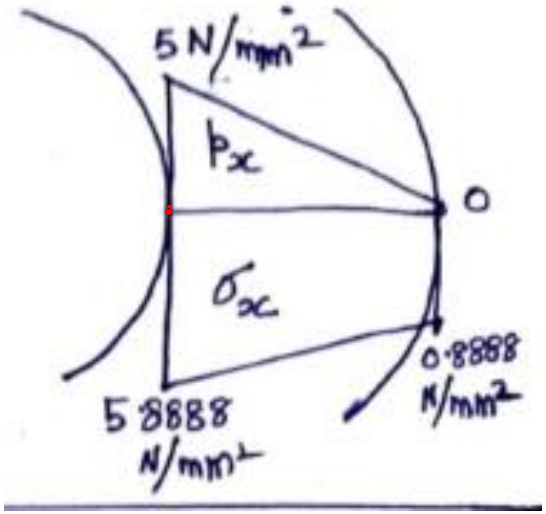
$$\sigma_{x=r_2} = \frac{b}{x^2} + a = \frac{2177.78}{20^2} + 0.4444$$

$$= 5.8888\ N/mm^2$$

$$\sigma_{x=r_1} = \frac{2177.78}{70^2} + 0.4444$$

$$= 0.8888\ N/mm^2$$

Stress distribution



Thick cylinder problem-2

- Find the thickness of metal necessary for a cylindrical shell of internal diameter 16 m to withstand an internal pressure of 8 N/mm^2. The maximum hoop stress in the section is not to exceed 35 N/mm^2.

Formula used:

Radial pressure

(max. at inner radius and zero at outer radius)

$$p_x = \frac{b}{x^2} - a$$

Hoop stress

(max. at inner radius and min. at outer radius)

$$\sigma_x = \frac{b}{x^2} + a$$

Thick sphere – Problem-3

- A spherical shell of 60 mm inside diameter has to withstand an internal pressure of 25 MPa. Find the thickness of the shell if the maximum tensile stress is to be 75 MPa.

Formula used:

Radial pressure

(max. at inner radius and zero at outer radius)

$$p_x = \frac{2b}{x^3} - a$$

Hoop stress

(max. at inner radius and min. at outer radius)

$$\sigma_x = \frac{b}{x^3} + a$$

Thick sphere – Problem-3 contd..

(9) Thick spherical shell

$r_2 = \frac{60}{2} = 30\,mm$, $p = 25\,MPa$

$= 25\,N/mm^2$

Radial pressure

$$p_x = \frac{2b}{x^3} - a$$

at $x = r_2$, $p_x = 25\,N/mm^2$

$$25 = \frac{2b}{30^3} - a$$

$$25 = \frac{2b}{27000} - a \quad \text{——— (1)}$$

Circumferential stress is maximum at inner radius at $x = r_2$, $\sigma_x = 75\,N/mm^2$

$$\sigma_x = \frac{b}{x^3} + a$$

$$75 = \frac{b}{30^3} + a$$

$$75 = \frac{b}{27000} + a \quad \text{——— (2)}$$

$$(1)+(2) \Rightarrow 100 = \frac{2b}{27000} + \frac{b}{27000}$$

$b = 900000$
$a = 41.67$

Thick sphere – Problem-3 contd..

Radial pressure is zero at outter radius.

at $x = r_1$, $p_x = 0$.

$$0 = \frac{2 \times 900000}{x^3} - 41.67$$

$r_1 = x = 35.08$

thickness $= r_1 - r_2 = 35.08 - 30$

$= 5.08$ mm.

Thick cylinder – Problem-4

7) Determine the maximum and minimum hoop stress across the section of a pipe of 400 mm internal diameter and 100 mm thick, when the pipe contains a fluid at a pressure of 8 N/mm^2. Also sketch the radial pressure distribution and hoop stress distribution across the section. (May 2017) (Nov/Dec 2017)

Formula used:

Radial pressure

(max. at inner radius and zero at outer radius)

$$p_x = \frac{b}{x^2} - a$$

Hoop stress

(max. at inner radius and min. at outer radius)

$$\sigma_x = \frac{b}{x^2} + a$$

Thick cylinder – Problem-4

1. At $x = r_1 = 200$ mm, $p_x = 8$ N/mm^2

2. At $x = r_2 = 300$ mm, $p_x = 0$

Substituting these boundary conditions in equation(i), we get

$$8 = \frac{b}{200^2} - a = \frac{b}{40000} - a \qquad \text{...(ii)}$$

and

$$0 = \frac{b}{300^2} - a = \frac{b}{90000} - a \qquad \text{...(iii)}$$

subtracting equation (iii) from equation (ii), we get

$$8 = \frac{b}{40000} - \frac{b}{90000} = \frac{9b - 4b}{360000} = \frac{5b}{360000}$$

$$b = \frac{360000 \times 8}{5} = 5760000$$

Substituting this value in equation (iii), we get

$$0 = \frac{5760000}{90000} - a \quad \text{or} \quad a = \frac{5760000}{90000} = 6.4$$

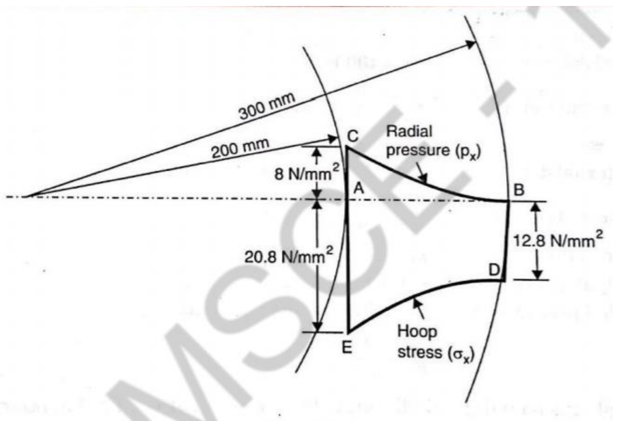
The values of 'a' and 'b' are substituted in the hoop stress.

Now hoop stress at any radius x is given by equation (18.2) as

$$\sigma_x = \frac{b}{x^2} + a = \frac{576000}{x^2} + 6.4$$

At x = 200 mm, $\sigma_{200} = \frac{576000}{200^2} + 6.4 = 14.4 + 6.4 = 20.8 \text{N/mm}^2$. Ans.

At x = 300 mm, $\sigma_{300} = \frac{576000}{300^2} + 6.4 = 6.4 + 6.4 = 12.8 \text{N/mm}^2$. Ans.



Revision problem-1

A thin cylinder of internal diameter 1.25 m contains a fluid at an internal pressure of 2 N/mm^2. Determine the maximum thickness of the cylinder if:

i) The longitudinal stress is not to exceed 30 N/mm^2

ii) The circumferential stress is not to exceed 45 N/mm^2

Hints:

- **Circumferential or hoop stress**

$\sigma_c = \frac{pd}{2t}$ **(calculate t)**

- **Longitudinal or axial stress**

$\sigma_l = \frac{pd}{4t}$ **(calculate t)**

Take the larger of two values of t

Revision problem-2

A cylindrical thin drum 80 cm in diameter and 3 m long has a shell thickness of 1 cm. if the drum is subjected to an internal pressure of 2.5 N/mm^2, determine

i) Change in diameter

ii) Change in length

iii) Change in volume

Take E = 2 x 10^5 N/mm^2 and Poisson's ratio = 0.25

Hints:

- Change in diameter

$$\delta d = \frac{pd}{2tE}\left(1 - \frac{\mu}{2}\right) \times d = \frac{pd^2}{2tE}\left(1 - \frac{\mu}{2}\right)$$

- Change in length

$$\delta l = \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right) \times l = \frac{pdl}{2tE}\left(\frac{1}{2} - \mu\right)$$

- Change in volume

$$\delta V = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right) \times V = \frac{pd}{2tE}\left(\frac{5}{2} - 2\mu\right) \times \frac{\pi}{4}\mathrm{d}^2 l$$

Revision problem-3

A spherical vessel 1.5 m diameter is subjected to an internal pressure of 2 N/mm^2. Find the thickness of the plate required if maximum stress is not to exceed 150 N/mm^2 and joint efficiency is 75%.

Hints:

$$\sigma_c = \frac{pd}{4t\eta}$$

Revision problem-4

A spherical shell of internal diameter 0.9 m and thickness 10 mm is subjected to an internal pressure of 1.4 N/mm^2. Determine the increase in diameter and increase in volume. Take E = 2 x 10^5 N/mm^2 and m = 3

Hints:

- Change in diameter
- $\delta d = \frac{\mathrm{pd^2}}{\mathrm{4tE}}(1-\mu)$
- Volumetric strain
- $\varepsilon_v = \frac{\delta V}{V} = \frac{\mathrm{3pd}}{\mathrm{4tE}}(1-\mu)$
- $\delta V = \frac{\mathrm{3pd}}{\mathrm{4tE}}(1-\mu) \times \textit{Volume of sphere}$
- Note: Poisson's ratio $\mu = \frac{1}{m}$

Revision problem-5

Find the thickness of metal necessary for a cylindrical shell of internal diameter 150 mm to withstand an internal pressure of 50 N/mm^2. The maximum hoop stress in the section is not to exceed 150 N/mm^2.

Hints

- Radial pressure

$$p_x = \frac{b}{x^2} - a$$

- Hoop stress

$$\sigma_x = \frac{b}{x^2} + a$$

- Where a and b are constants


THANK
YOU